First Semester/ Mathematics

Contents:

References:

Calculus Volume 2 (2017):

https://d3bxy9euw4e147.cloudfront.net/oscms-prodcms/media/documents/CalculusVolume2-OP.pdf

Thomas Calculus (FOURTEENTH EDITION):
https://rodrigopacios.github.io/mrpacios/download/Thomas_Calculus.pdf

1. Conical Sections

1.1 Introduction

Conic sections get their name because they can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called nappes. One nappe is what most people mean by "cone," having the shape of a party hat. A right circular cone can be generated by revolving a line passing through the origin around the y-axis.

Conic sections are generated by the intersection of a plane with a cone (Figure 1.1). If the plane is parallel to the axis of revolution (they-axis), then the conic section is a **hyperbola**. If the plane is parallel to the generating line, the conic section is a **parabola**. If the plane is perpendicular to the axis of revolution, the conic section is a **circle**. If the plane intersects one nappe at an angle to the axis (other than 90°), then the conic section is an **ellipse**.

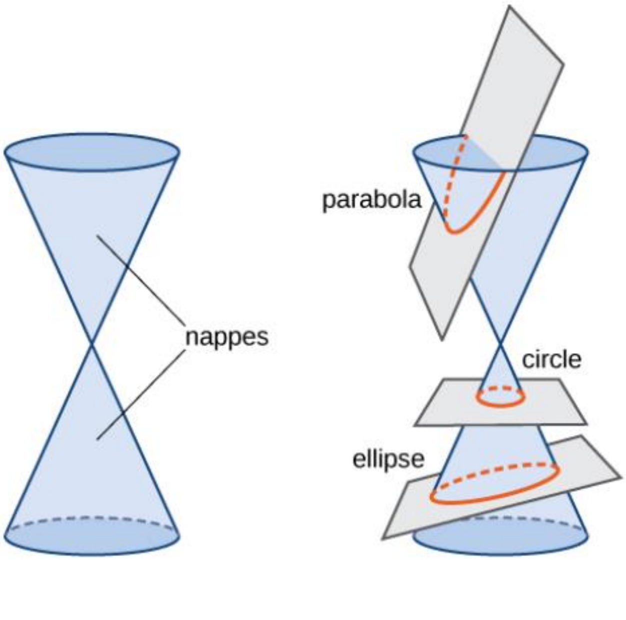


Figure 1.1 The four conic sections. Each conic is determined by the angle the plane makes with the axis of the cone

1.2 Parabolas

A parabola is the set of all points whose distance from a fixed point, called the focus, is equal to the distance from a fixed line, called the directrix. The point halfway between the focus and the directrix is called the vertex of the parabola.

Given a parabola opening upward with vertex located at (h, k) and focus located at (h, k+ p), where p is a constant, the equation for the parabola is given by

$$y = \frac{1}{4p}(x - h)^2 + k. \tag{1.1}$$

This is the standard form of a parabola.

We can also study the cases when the parabola opens down or to the left or the right. The equation for each of these cases can also be written in standard form as shown in the following graphs.

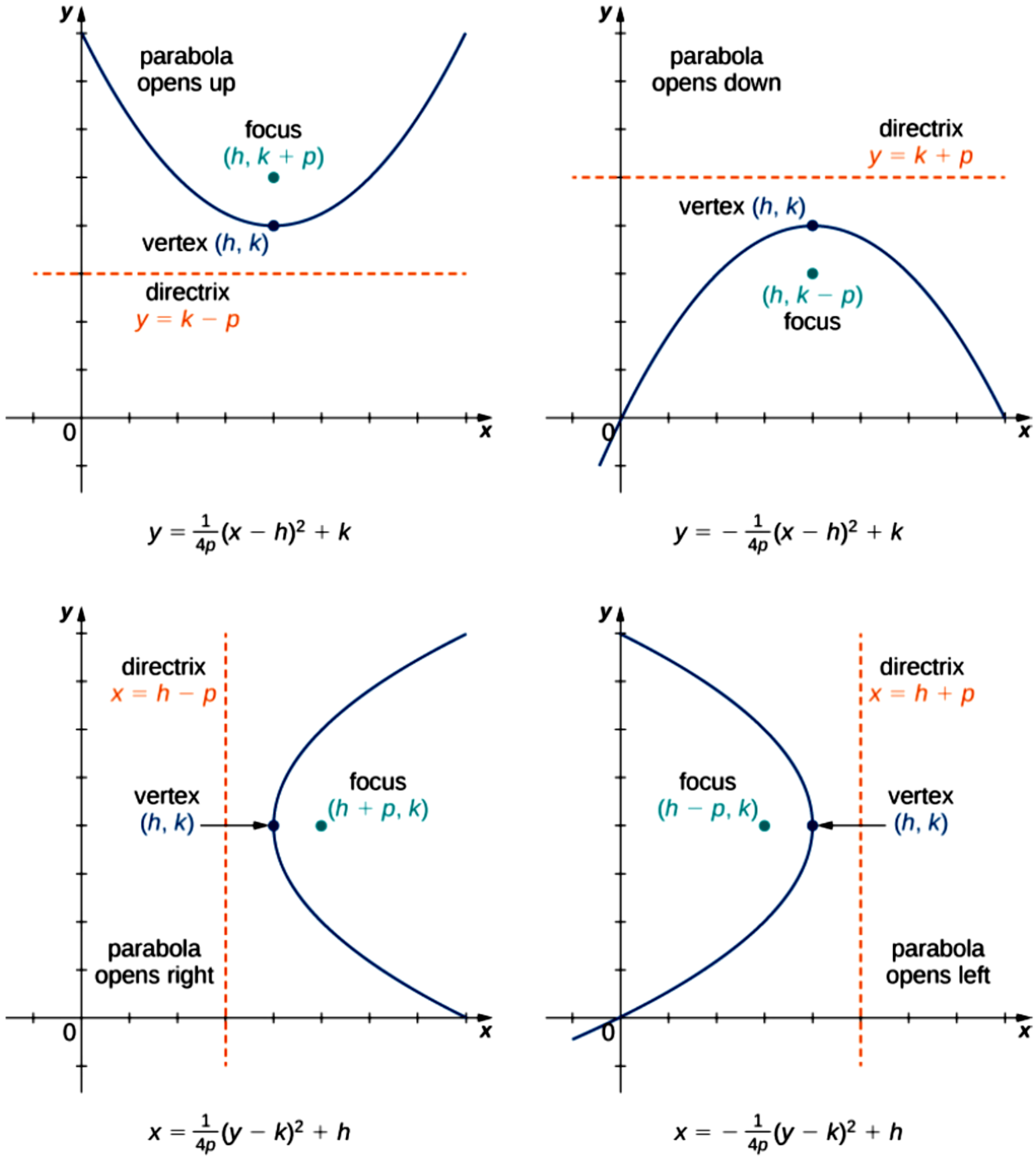


Figure 1.2 Four parabolas, opening in various directions, along with their equations in standard form.

Example 1.1

Put the equation $x^2 - 4x - 8y + 12 = 0$ into standard form and graph the resulting parabola.

Solution:

$8y = x^2 - 4x + 12$

$8y = (x^2 - 4x) + 12$

$8y = (x^2 - 4x + 4) + 12 - 4$

$y = 1/8\ (x - 2)^2 + 1$

This equation is now in standard form. Comparing this to Equation 1.1 gives h=2, k=1, and p=2. The parabola opens up, with vertex at (2, 1), focus at (2, 3), and directrix y=−1. The graph of this parabola appears as follows.

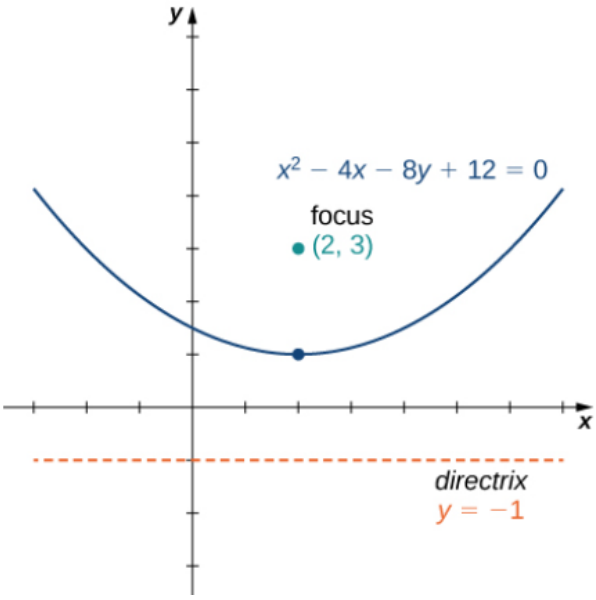


Figure 1.3 The parabola in Example 1.1.

1.3 Ellipses

An ellipse is the set of all points for which the sum of their distances from two fixed points (the foci) is constant.

Consider the ellipse with center (h, k), a horizontal major axis with length 2a, and a vertical minor axis with length 2b. Then the equation of this ellipse in standard form is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

and the foci are located at (h±c, k), where $c^2 = a^2 - b^2$. The equations of the directrices are $x = h \pm a^2/c$. If the major axis is vertical, then the equation of the ellipse becomes

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

and the foci are located at (h, k±c), where $c^2 = a^2 - b^2$. The equations of the directrices in this case are $y = k \pm a^2/c$.

Example 1.2

Put the equation $9x^2+4y^2-36x+24y+36=0$ into standard form and graph the resulting ellipse.

Solution:

$9x^2+4y^2-36x+24y=-36$

$(9x^2-36x) + (4y^2+24y) = -36$

$9(x^2-4x) +4(y^2+6y) = -36$

$9(x^2-4x+4) +4(y^2+6y+9) = -36+36+36$

$9(x^2-4x+4)+4(y^2+6y+9)=36$

$$\begin{aligned} 9(x-2)^2 + 4(y+3)^2 &= 36 \\ \frac{9(x-2)^2}{36} + \frac{4(y+3)^2}{36} &= 1 \\ \frac{(x-2)^2}{4} + \frac{(y+3)^2}{9} &= 1. \end{aligned}$$

The equation is now in standard form. Comparing this to standard equation gives h=2, k =−3, a=3, and b=2. This is a vertical ellipse with center at (2,−3), major axis 6, and minor axis 4. The graph of this ellipse appears as follows.

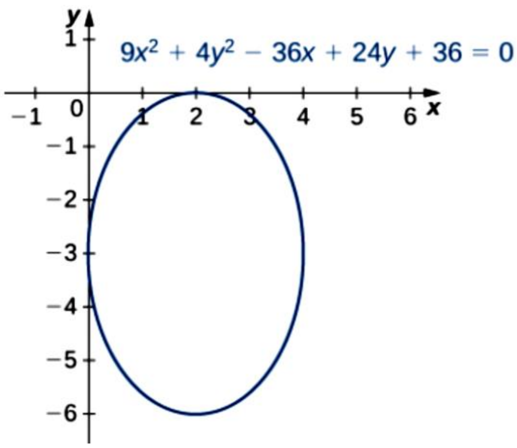


Figure 1.4 The ellipse in Example 1.2.

1.4 Hyperbolas

A hyperbola is the set of all points where the difference between their distances from two fixed points (the foci) is constant.

Consider the hyperbola with center (h, k), a horizontal major axis, and a vertical minor axis. Then the equation of this ellipse is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

and the foci are located at (h±c, k), where $c^2=a^2+b^2$. The equations of the asymptotes are given by y=k±b/a(x−h). The equations of the directrices are

$$x=k\pm\frac{a^2}{\sqrt{a^2+b^2}}=h\pm\frac{a^2}{c}.$$

If the major axis is vertical, then the equation of the hyperbola becomes

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

and the foci are located at (h, k±c), where $c^2=a^2+b^2$. The equations of the asymptotes are given by y=k±a/b(x−h). The equations of the directrices are

$$y=k\pm\frac{a^2}{\sqrt{a^2+b^2}}=k\pm\frac{a^2}{c}.$$

Example 1.3

Put the equation $9x^2-16y^2+36x+32y-124=0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Solution:

$9x^2-16y^2+36x+32y=124$

$(9x^2+36x)-(16y^2-32y) = 124$

$9(x^2+4x)-16(y^2-2y) = 124$

$9(x^2+4x+4)-16(y^2-2y+1)=124+36-16$

$9(x^2+4x+4)-16(y^2-2y+1)=144$

$$9(x+2)^2 - 16(y-1)^2 = 144$$
$$\frac{9(x+2)^2}{144} - \frac{16(y-1)^2}{144} = 1$$
$$\frac{(x+2)^2}{16} - \frac{(y-1)^2}{9} = 1.$$

The equation is now in standard form. Comparing this to standard equation gives h=−2, k =1, a=4, and b=3. This is a horizontal hyperbola with center at (−2, 1) and asymptotes given by the equations y=1±3/4(x+2). The graph of this hyperbola appears in the following figure.

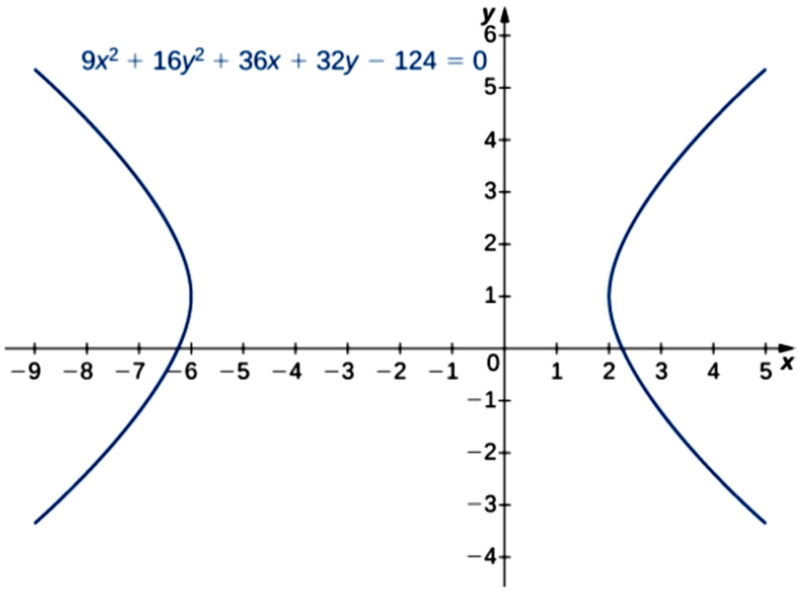


Figure 1.5 Graph of the hyperbola in Example 1.3.

1.5 Eccentricity and Directrix

The eccentricity e of a conic section is defined to be the distance from any point on the conic section to its focus, divided by the perpendicular distance from that point to the nearest directrix. This value is constant for any conic section, and can define the conic section as well:

1. If e=1, the conic is a parabola.

2. If e<1, it is an ellipse.

e= c/a, where a>c

3. If e>1, it is a hyperbola.

e= c/a, where a<c

The eccentricity of a circle is zero. The directrix of a conic section is the line that, together with the point known as the focus, serves to define a conic section. Hyperbolas and noncircular ellipses have two foci and two associated directrices. Parabolas have one focus and one directrix.

The three conic sections with their directrices appear in the following figure.

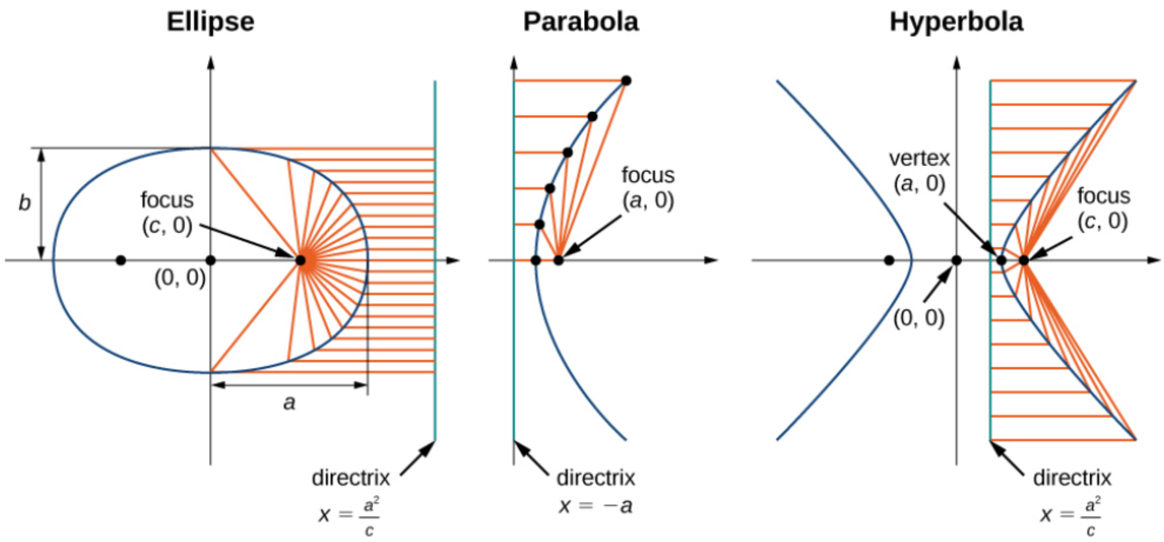


Figure 1.5 The three conic sections with their foci and directrices.

Example 1.4

Determine the eccentricity of the ellipse described by the equation

$$\frac{(x-3)^2}{16}+\frac{(y+2)^2}{25}=1.$$

Solution:

From the equation we see that a=5 and b=4. The value of c can be calculated using the equation $a^2 = b^2 + c^2$ for an ellipse. Substituting the values of a and b and solving for c gives c=3. Therefore the eccentricity of the ellipse is e= c/a = 3/5 =0.6.

Example 1.5

Find a Cartesian equation for the hyperbola centered at the origin that has a focus at (3, 0) and the line x = 1 as the corresponding directrix.

Solution:

We first use the dimensions shown in Figure to find the hyperbola's eccentricity. The focus is (see Figure)

(c, 0) = (3, 0), so c = 3.

Again from Figure, the directrix is the line

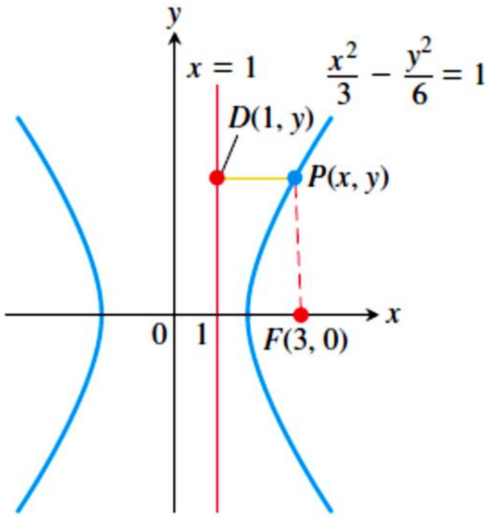
x = a/e = 1, so a = e.

When combined with the equation e = c/a that defines eccentricity, these results give

$$e = \frac{c}{a} = \frac{3}{e}, \quad \text{so} \quad e^2 = 3 \quad \text{and} \quad e = \sqrt{3}.$$

Knowing e, we can now derive the equation we want from the equation PF = e.PD. In the coordinates of Figure, we have

$$\begin{aligned} PF &= e \cdot PD \\ \sqrt{(x-3)^2 + (y-0)^2} &= \sqrt{3}\,|x-1| \\ x^2 - 6x + 9 + y^2 &= 3(x^2 - 2x + 1) \\ 2x^2 - y^2 &= 6 \\ \frac{x^2}{3} - \frac{y^2}{6} &= 1. \end{aligned}$$



1.6 Polar Equations of Conic Sections

Sometimes it is useful to write or identify the equation of a conic section in polar form. To do this, we need the concept of the focal parameter. The focal parameter of a conic section p is defined as the distance from a focus to the nearest directrix. The following table gives the focal parameters for the different types of conics, where a is the length of the semi-major axis (i.e., half the length of the major axis), c is the distance from the origin to the focus, and e is the eccentricity. In the case of a parabola, a represents the distance from the vertex to the focus.

Conic	e	p
Ellipse	$0 < e < 1$	$\frac{a^2 - c^2}{c} = \frac{a(1 - e^2)}{c}$
Parabola	$e = 1$	$2a$
Hyperbola	$e > 1$	$\frac{c^2 - a^2}{c} = \frac{a(e^2 - 1)}{e}$

Table 1.1 Eccentricities and Focal Parameters of the Conic Sections

The polar equation of a conic section with focal parameter p is given by

$$r = \frac{ep}{1 \pm e\cos\theta} \text{ or } r = \frac{ep}{1 \pm e\sin\theta}.$$

Note: You may see variations of Equation, depending on the location of the directrix.

1. For the directrix is the line x = p

$$r = \frac{ep}{1 + e\ cos\theta}$$

2. For the directrix is the line x = -p

$$r = \frac{ep}{1 - e\ cos\theta}$$

3. For the directrix is the line y = p

$$r = \frac{ep}{1 + e\ sin\theta}$$

4. For the directrix is the line y = -p

$$r = \frac{ep}{1 - e\ sin\theta}$$

Example 1.6

Find an equation for the hyperbola with eccentricity 3/2 and directrix x = 2. Where focus at the origin.

Solution:

When focus at the origin p = 2 and e = 3/2:

$$r = \frac{2(3/2)}{1 + (3/2)\cos\theta} \quad \text{or} \quad r = \frac{6}{2 + 3\cos\theta}.$$

Example 1.7

Identify and create a graph of the conic section described by the equation

$$r = \frac{3}{1 + 2\cos\theta}.$$

Solution:

The constant term in the denominator is 1, so the eccentricity of the conic is 2. This is a hyperbola. The focal parameter p can be calculated by using the equation ep=3. Since e=2, this gives p= 3/2. The cosine function appears in the denominator, so the hyperbola is horizontal. Pick a few values for θ and create a table of values. Then we can graph the hyperbola (Figure 1.6).

θ	r	θ	r
0	1	π	-3
$\frac{\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$	$\frac{5\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$
$\frac{\pi}{2}$	3	$\frac{3\pi}{2}$	3
$\frac{3\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$	$\frac{7\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$

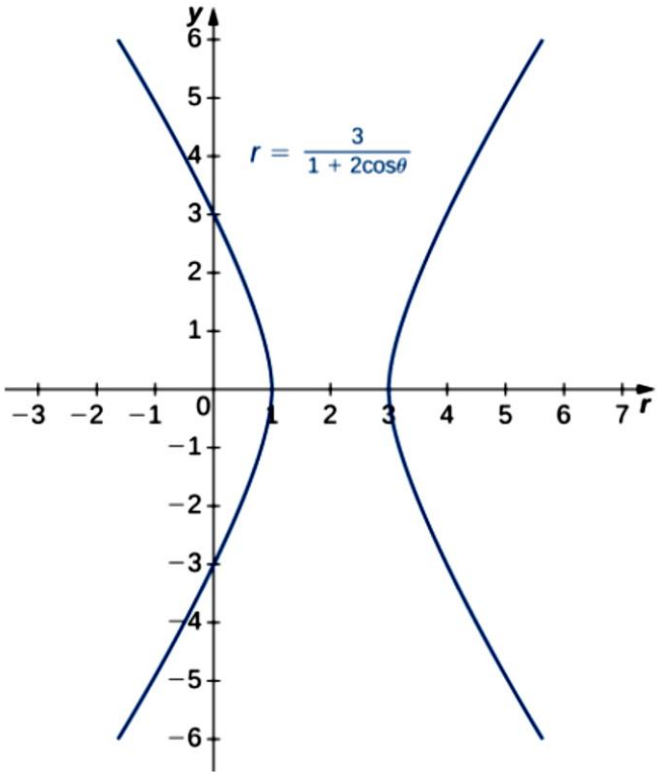


Figure 1.6 Graph of the hyperbola described in Example 1.5.

1.7 General Equations of Degree Two

A general equation of degree two can be written in the form

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

The graph of an equation of this form is a conic section. If $B\neq 0$ then the coordinate axes are rotated. To identify the conic section, we use the discriminant of the conic section $4AC-B^2$. One of the following cases must be true:

1. $4AC-B^2>0$. If so, the graph is an ellipse.

2. $4AC-B^2=0$. If so, the graph is a parabola.

3. $4AC-B^2<0$. If so, the graph is a hyperbola.

To determine the angle θ of rotation of the conic section, we use the formula $\cot 2\theta = (A-C)/B$. In this case $A=C=0$ and $B=1$, so $\cot 2\theta=(0-0)/1=0$ and $\theta=45°$. The method for graphing a conic section with rotated axes involves determining the coefficients of the conic in the rotated coordinate system. The new coefficients are labeled A′, B′, C′, D′, E′, and F′, and are given by the formulas

$A' = A\cos^2\theta+B\cos\theta\sin\theta+C\sin^2\theta$

$B' = 0$

$C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta$

$D' = D\cos\theta + E\sin\theta$

$E' = -D\sin\theta + E\cos\theta$

$F' = F.$

The procedure for graphing a rotated conic is the following:

1. Identify the conic section using the discriminant $4AC-B^2$.
2. Determine θ using the formula $\cot 2\theta = (A-C)/B$.
3. Calculate A', B', C', D', E', and F'.
4. Rewrite the original equation using A', B', C', D', E', and F'.
5. Draw a graph using the rotated equation.

Example 1.7

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

$13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0.$

Solution:

In this equation, $A = 13, B = -6\sqrt{3}, C = 7, D = 0, E = 0,$ and $F = -256.$ The discriminant of this equation is $4AC - B^2 = 4(13)(7) - \left(-6\sqrt{3}\right)^2 = 364 - 108 = 256.$ Therefore this conic is an ellipse. To calculate the angle of rotation of the axes, use $\cot 2\theta = \frac{A-C}{B}.$ This gives

$$\begin{aligned}\cot 2\theta &= \frac{A-C}{B}\\ &= \frac{13-7}{-6\sqrt{3}}\\ &= -\frac{\sqrt{3}}{3}.\end{aligned}$$

Therefore $2\theta = 120^{\circ}$ and $\theta = 60^{\circ}$, which is the angle of the rotation of the axes.

To determine the rotated coefficients, use the formulas given above:

$$\begin{aligned}A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta\\ &= 13\cos^2 60 + \left(-6\sqrt{3}\right)\cos 60\sin 60 + 7\sin^2 60\\ &= 13\left(\frac{1}{2}\right)^2 - 6\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 7\left(\frac{\sqrt{3}}{2}\right)^2\\ &= 4,\end{aligned}$$

$$
\begin{aligned}
B' &= 0, \\
C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\
&= 13\sin^2 60 + \left(-6\sqrt{3}\right)\sin 60\cos 60 = 7\cos^2 60 \\
&= \left(\frac{\sqrt{3}}{2}\right)^2 + 6\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 7\left(\frac{1}{2}\right)^2 \\
&= 16, \\
D' &= D\cos\theta + E\sin\theta \\
&= (0)\cos 60 + (0)\sin 60 \\
&= 0, \\
E' &= -D\sin\theta + E\cos\theta \\
&= -(0)\sin 60 + (0)\cos 60 \\
&= 0, \\
F' &= F \\
&= -256.
\end{aligned}
$$

The equation of the conic in the rotated coordinate system becomes

$$
\begin{aligned}
4(x')^2 + 16(y')^2 &= 256 \\
\frac{(x')^2}{64} + \frac{(y')^2}{16} &= 1.
\end{aligned}
$$

A graph of this conic section appears as follows.

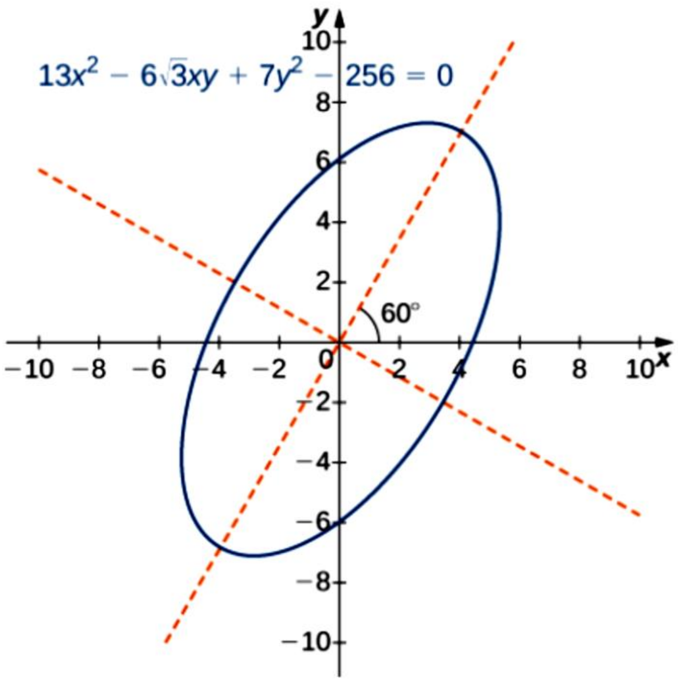


Figure 1.7 Graph of the ellipse described by the equation

2. Parametric Equations

2.1 Introduction

In this section we examine parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then x and y are defined as functions of time, and (x(t), y(t))can describe the position in the plane of a given object as it moves along a curved path.

2.2 Parametric Equations and Their Graphs

If x and y are continuous functions of ton an interval I, then the equations x = x(t) and y= y(t) are called parametric equations and t is called the parameter. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a parametric curve or plane curve, and is denoted by C.

<u>Example 2.1</u>

Sketch the curve described by the following parametric equations:

a. $x(t) = t - 1, \quad y(t) = 2t + 4, \quad -3 \le t \le 2$

b. $x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad -2 \le t \le 3$

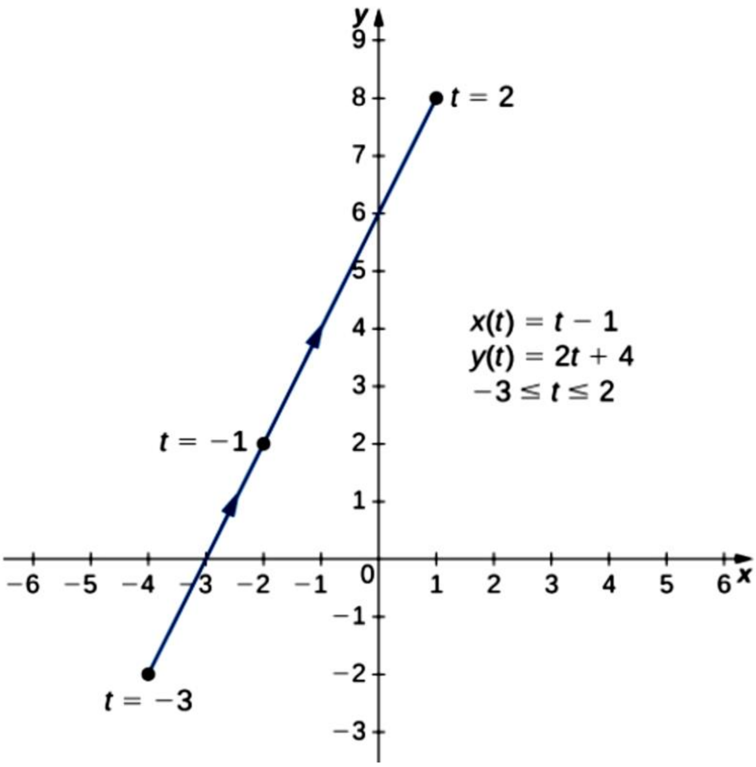
c. $x(t) = 4\cos t, \quad y(t) = 4\sin t, \quad 0 \le t \le 2\pi$

Solution:

a. To create a graph of this curve, first set up a table of values. Since the independent variable in both x(t) and y(t) is t, let t appear in the first column. Then x(t) and y(t) will appear in the second and third columns of the table.

t	$x(t)$	$y(t)$
−3	−4	−2
−2	−3	0
−1	−2	2
0	−1	4
1	0	6
2	1	8

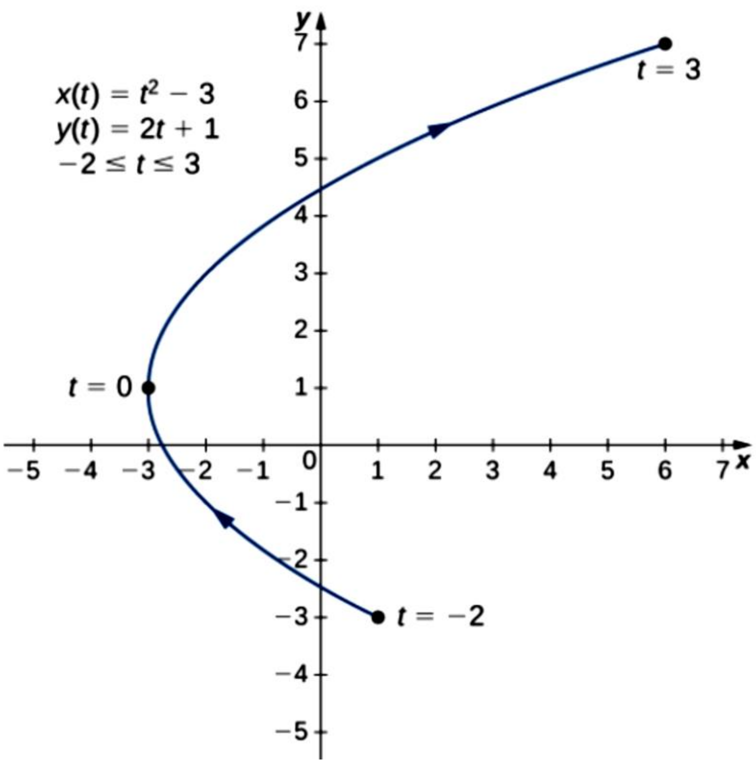
The graph of these points appears in Figure. The arrows on the graph indicate the orientation of the graph, that is, the direction that a point moves on the graph as t varies from −3 to 2.



b. To create a graph of this curve, again set up a table of values.

t	$x(t)$	$y(t)$
−2	1	−3
−1	−2	−1
0	−3	1
1	−2	3
2	1	5
3	6	7

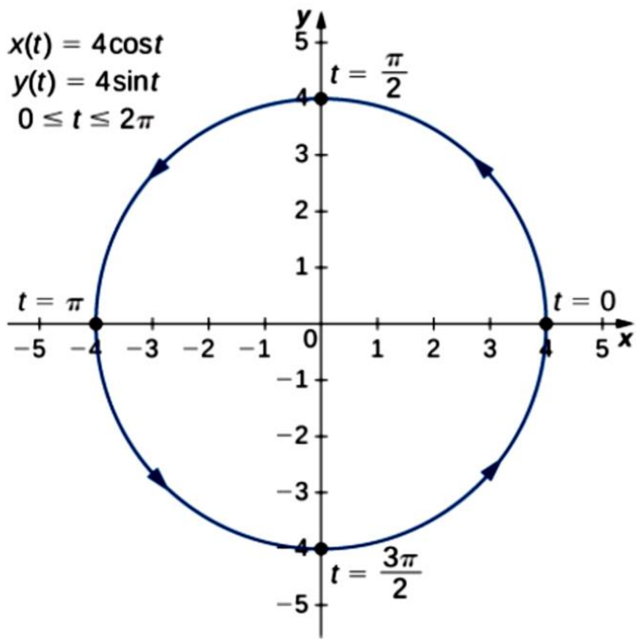
The graph of this plane curve appears in the following graph.



c. In this case, use multiples of $\pi/6$ fortand create another table of values:

t	$x(t)$	$y(t)$		t	$x(t)$	$y(t)$
0	4	0		$\frac{7\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2		$\frac{4\pi}{3}$	-2	$-2\sqrt{3} \approx -3.5$
$\frac{\pi}{3}$	2	$2\sqrt{3} \approx 3.5$		$\frac{3\pi}{2}$	0	-4
$\frac{\pi}{2}$	0	4		$\frac{5\pi}{3}$	2	$-2\sqrt{3} \approx -3.5$
$\frac{2\pi}{3}$	-2	$2\sqrt{3} \approx 3.5$		$\frac{11\pi}{6}$	$2\sqrt{3} \approx 3.5$	2
$\frac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2		2π	4	0
π	-4	0				

The graph of this plane curve appears in the following graph.



Example 2.2

The position P(x, y) of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = \sqrt{t}, \qquad y = t, \qquad t \geq 0.$$

Identify the path traced by the particle and describe the motion.

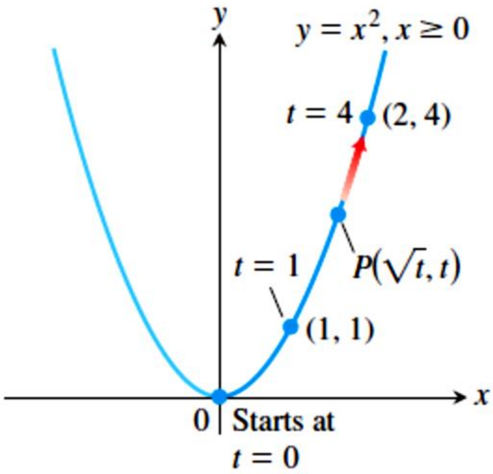
Solution:

We try to identify the path by eliminating t between the equations $x = \sqrt{t}$ and y = t, which might produce a re-cognizable algebraic relation between x and y. We find that

$$y = t = (\sqrt{t})^2 = x^2.$$

Thus, the particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along the parabola $y = x^2$.

The particle starts at (0, 0) when t = 0 and rises into the first quadrant as t increases (Figure). The parameter interval is $[0, \infty)$ and there is no terminal point.



Example 2.2

Sketch and identify the path traced by the point P(x, y) if

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}, \qquad t > 0.$$

Solution:

Taking the difference between x and y as given by the parametric equations, we find that

$$x - y = \left(t + \frac{1}{t}\right) - \left(t - \frac{1}{t}\right) = \frac{2}{t}.$$

If we add the two parametric equations, we get

$$x + y = \left(t + \frac{1}{t}\right) + \left(t - \frac{1}{t}\right) = 2t.$$

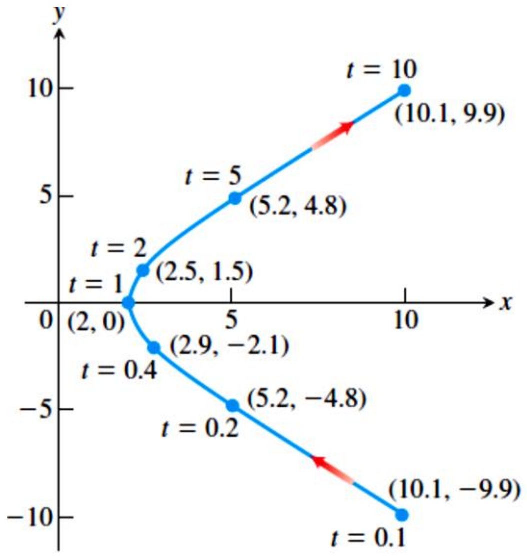
We can then eliminate the parameter *t* by multiplying these last equations together:

$$(x - y)(x + y) = \left(\frac{2}{t}\right)(2t) = 4.$$

Expanding the expression on the left-hand side, we obtain a standard equation for a hyperbola:

$x^2 - y^2 = 4.$

There are points (x, y) on the hyperbola that do not satisfy the parametric equation $x = t + (1/t)$, $t > 1$.



2.3 Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables x and y. Then we can apply any previous knowledge of equations of curves in the plane to identify the curve.

Example 2.2

Eliminate the parameter for each of the plane curves described by the following parametric equations and describe the resulting graph.

a. $x(t) = \sqrt{2t + 4}, \quad y(t) = 2t + 1, \quad -2 \le t \le 6$

b. $x(t) = 4\cos t, \quad y(t) = 3\sin t, \quad 0 \le t \le 2\pi$

Solution:

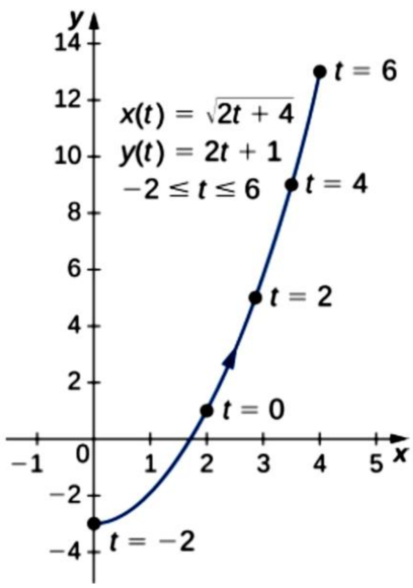
a. To eliminate the parameter, we can solve either of the equations for t. For example, solving the first equation for t gives

$$\begin{aligned} x &= \sqrt{2t+4} \\ x^2 &= 2t+4 \\ x^2-4 &= 2t \\ t &= \frac{x^2-4}{2}. \end{aligned}$$

Note that when we square both sides it is important to observe that $x \geq 0$. Substituting $t = \dfrac{x^2-4}{2}$ this into $y(t)$ yields

$$\begin{aligned} y(t) &= 2t+1 \\ y &= 2\left(\frac{x^2-4}{2}\right)+1 \\ y &= x^2-4+1 \\ y &= x^2-3. \end{aligned}$$

This is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter t. When $t=-2$, $x=\sqrt{2(-2)+4}=0$, and when $t=6$, $x=\sqrt{2(6)+4}=4$. The graph of this plane curve follows.



b. Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

$x(t)=4\cos t$ and $y(t)=3\sin t$.

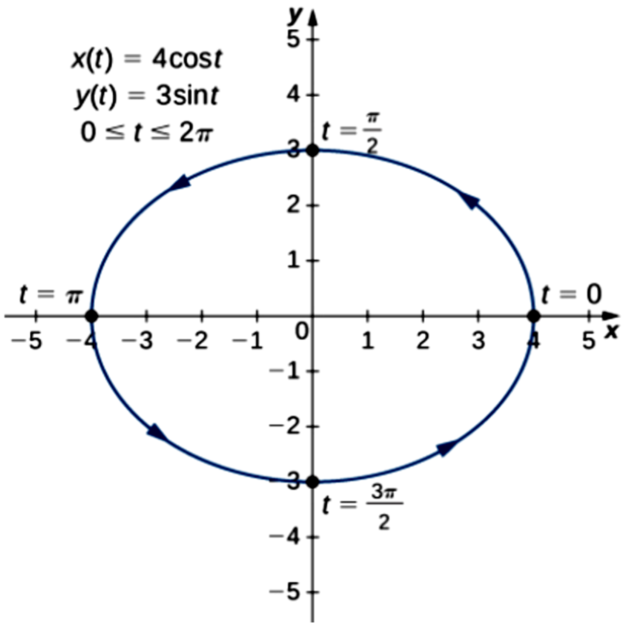
Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the t) gives us

$\cos t = x/4$ and $\sin t = y/3$.

Now use the Pythagorean identity $\cos^2 t+\sin^2 t =1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of x and y. This gives

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

This is the equation of a horizontal ellipse centered at the origin, with semimajor axis 4 and semiminor axis 3 as shown in the following graph.



Example 2.3

Find two different pairs of parametric equations to represent the graph of $y=2x^2-3$.

Solution:

First, it is always possible to parameterize a curve by defining $x(t)=t$, then replacing x with t in the equation for y(t). This gives the parameterization

$x(t)=t,\ y(t)=2t^2-3.$

Since there is no restriction on the domain in the original graph, there is no restriction on the values of t. We have complete freedom in the choice for the second parameterization. For example, we can choose $x(t)=3t-2$. The only thing we need to check is that there are no restrictions imposed on x; that is, the range of x(t) is all real numbers. This is the case for $x(t)=3t-2$. Now since $y=2x^2-3$, we can substitute $x(t)=3t-2$ for x. This gives

$y(t) =2(3t-2)2-2$

$=2(9t^2-12t+4)-2$

$=18t^2-24t+8-2$

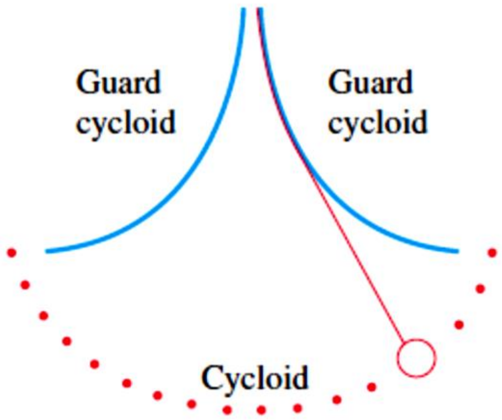
$=18t^2-24t+6.$

Therefore, a second parameterization of the curve can be written as

$x(t)=3t-2$ and $y(t)=18t^2-24t+6.$

2.4 Cycloids

The problem with a pendulum clock whose bob swings in a circular arc is that the frequency of the swing depends on the amplitude of the swing. The wider the swing, the longer it takes the bob to return to center (its lowest position). This does not happen if the bob can be made to swing in a cycloid. In 1673, Christian Huygens designed a pendulum clock whose bob would swing in a cycloid. He hung the bob from a fine wire constrained by guards that caused it to draw up as it swung away from center (Figure). We describe the path parametrically in the next example.



Example 2.4

A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a cycloid.

Solution:

We take the line to be the x-axis, mark a point P on the wheel, start the wheel with P at the origin, and roll the wheel to the right. As parameter, we use the angle t through which the wheel turns, measured in radians. Figure shows the wheel a short while later when its base lies at units from the origin. The wheel's center C lies at (at, a) and the coordinates of P are

$x = at + a\cos u, \;\; y = a + a\sin u.$

To express u in terms of t, we observe that $t + u = 3\pi/2$ in the figure, so that

$u = 3\pi/2 - t.$

This makes

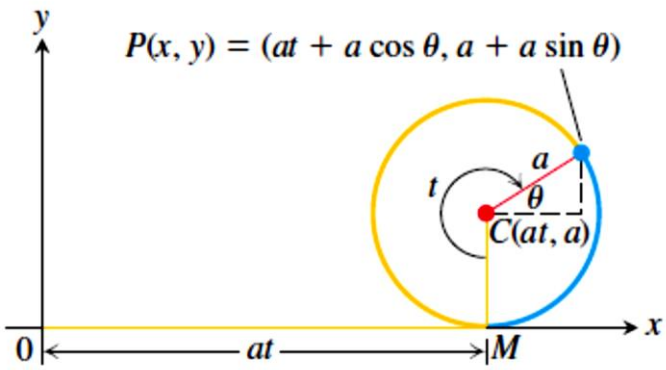
$\cos u = \cos(3\pi/2 - t) = -\sin t, \;\; \sin u = \sin(3\pi/2 - t) = -\cos t.$

The equations we seek are

$x = at - a\sin t, \;\; y = a - a\cos t.$

These are usually written with the a factored out: $x = a(t - \sin t), \;\; y = a(1 - \cos t).$

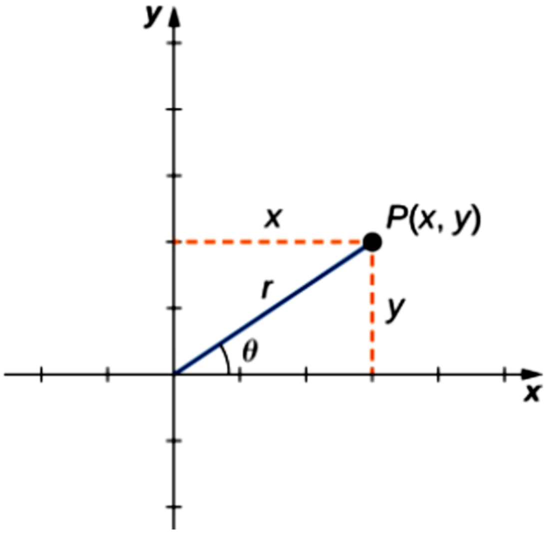
The Figure shows the first arch of the cycloid and part of the next.



3. Polar Coordinates

3.1 Defining Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider Figure. The point P has Cartesian coordinates (x, y). The line segment connecting the origin to the point P measures the distance from the origin to P and has length r. The angle between the positive x-axis and the line segment has measure θ. This observation suggests a natural correspondence between the coordinate pair (x, y) and the values r and θ. This correspondence is the basis of the polar coordinate system. Note that every point in the Cartesian plane has two values (hence the term ordered pair) associated with it. In the polar coordinate system, each point also two values associated with it: r and θ.



Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$x = r\cos\theta$ and

$y = r\sin\theta$,

$r^2 = x^2+y^2$ and

$\tan\theta = y/x$.

These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates.

Example 3.1

Convert each of the following points into polar coordinates.

a. (1, 1)

b. (−3, 4)

c. (0, 3)

d. $(5\sqrt{3}, -5)$

Convert each of the following points into rectangular coordinates.

e. $(3, \pi/3)$

f. $(2, 3\pi/2)$

g. $(6, -5\pi/6)$

Solution:

a. Use x =1 and y=1

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 1^2 + 1^2 \\ r &= \sqrt{2} \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{1}{1} = 1 \\ \theta &= \frac{\pi}{4}. \end{aligned}$$

Therefore this point can be represented as $(2, \pi/4)$ in polar coordinates.

b. Use x =−3 and y=4

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (-3)^2 + (4)^2 \\ r &= 5 \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= -\frac{4}{3} \\ \theta &= -\arctan\left(\frac{4}{3}\right) \\ &\approx 2.21. \end{aligned}$$

Therefore this point can be represented as (5, 2.21) in polar coordinates.

c. Use x =0 and y=3

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (3)^2 + (0)^2 \\ &= 9 + 0 \\ r &= 3 \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{3}{0}. \end{aligned}$$

Direct application of the second equation leads to division by zero. Graphing the point (0, 3) on the rectangular coordinate system reveals that the point is located on the positive y-axis. The angle between the positive x-axis and the positive y-axis is $\pi/2$. Therefore this point can be represented as $(3, \pi\ 2)$ in polar coordinates.

d. Use $x = 5\sqrt{3}$ and $y = -5$

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (5\sqrt{3})^2 + (-5)^2 \\ &= 75 + 25 \\ r &= 10 \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{-5}{5\sqrt{3}} = -\frac{\sqrt{3}}{3} \\ \theta &= -\frac{\pi}{6}. \end{aligned}$$

Therefore this point can be represented as $(10, -\pi/6)$ in polar coordinates.

e. Use $r = 3$ and $\theta = \pi/3$

$$\begin{aligned} x &= r\cos\theta \\ &= 3\cos\left(\frac{\pi}{3}\right) \\ &= 3\left(\frac{1}{2}\right) = \frac{3}{2} \end{aligned} \quad \text{and} \quad \begin{aligned} y &= r\sin\theta \\ &= 3\sin\left(\frac{\pi}{3}\right) \\ &= 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}. \end{aligned}$$

Therefore this point can be represented as $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$ in rectangular coordinates.

f. Use $r = 2$ and $\theta = 3\pi/2$

$$\begin{aligned} x &= r\cos\theta \\ &= 2\cos\left(\frac{3\pi}{2}\right) \\ &= 2(0) = 0 \end{aligned} \quad \text{and} \quad \begin{aligned} y &= r\sin\theta \\ &= 2\sin\left(\frac{3\pi}{2}\right) \\ &= 2(-1) = -2. \end{aligned}$$

Therefore this point can be represented as $(0, -2)$ in rectangular coordinates.

g. Use $r = 6$ and $\theta = -5\pi/6$

$$\begin{aligned} x &= r\cos\theta \\ &= 6\cos\left(-\frac{5\pi}{6}\right) \\ &= 6\left(-\frac{\sqrt{3}}{2}\right) \\ &= -3\sqrt{3} \end{aligned} \quad \text{and} \quad \begin{aligned} y &= r\sin\theta \\ &= 6\sin\left(-\frac{5\pi}{6}\right) \\ &= 6\left(-\frac{1}{2}\right) \\ &= -3. \end{aligned}$$

Therefore this point can be represented as $\left(-3\sqrt{3}, -3\right)$ in rectangular coordinates.

Example 3.2

Plot each of the following points on the polar plane.

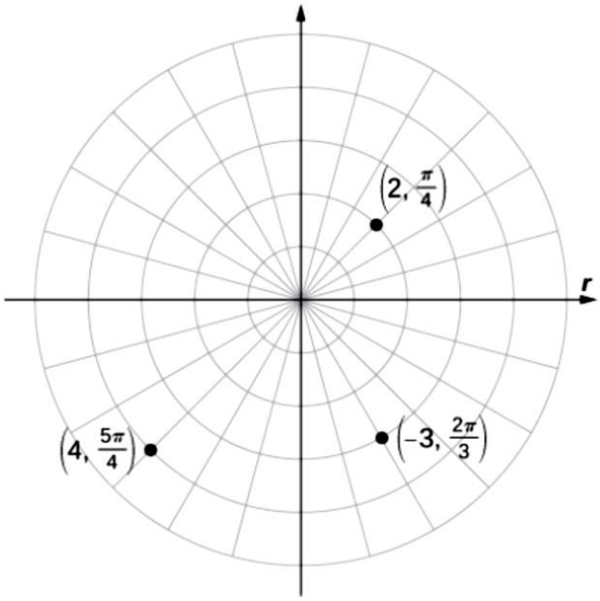
a. $(2, \pi/4)$

b. $(-3, 2\pi/3)$

c. $(4, 5\pi/4)$

Solution:

The three points are plotted in the following figure.



Example 3.3

Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

Solution:

We apply the equations relating polar and Cartesian coordinates:

$x^2 + (y - 3)^2 = 9$

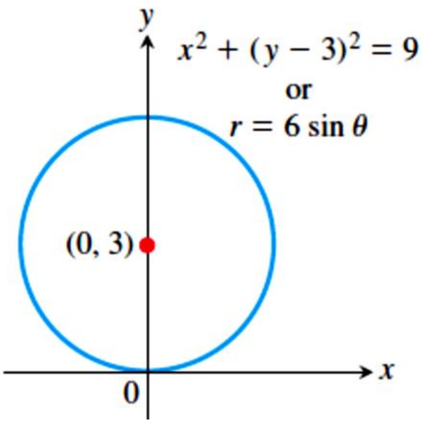
$x^2 + y^2 - 6y + 9 = 9$ Expand $(y - 3)^2$.

$x^2 + y^2 - 6y = 0$ Cancelation

$r^2 - 6r \sin u = 0$ $x^2 + y^2 = r^2$, $y = r \sin u$

$r = 0$ or $r - 6 \sin u = 0$

$r = 6 \sin u$ Includes both possibilities



3.2 Polar Curves

In the rectangular coordinate system, we can graph a function y= f(x) and create a curve in the Cartesian plane. In a similar fashion, we can graph a curve that is generated by a function r = f(θ).

To plotting a curve in polar coordinates should be:

1. Create a table with two columns. The first column is for θ, and the second column is for r.

2. Create a list of values for θ.

3. Calculate the corresponding r values for each θ.

4. Plot each ordered pair (r, θ) on the coordinate axes.

5. Connect the points and look for a pattern.

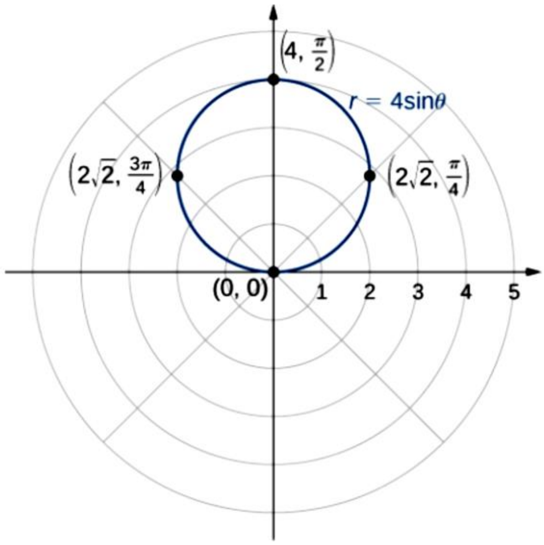
Example 3.3

Graph the curve defined by the function r = 4 sinθ. Identify the curve and rewrite the equation in rectangular coordinates.

Solution:

Because the function is a multiple of a sine function, it is periodic with period 2π, so use values for θ between 0 and 2π. The result of steps 1–3 appear in the following table.

θ	$r = 4\sin\theta$		θ	$r = 4\sin\theta$
0	0		π	0
$\frac{\pi}{6}$	2		$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$		$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$		$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{\pi}{2}$	4		$\frac{3\pi}{2}$	4
$\frac{2\pi}{3}$	$2\sqrt{3} \approx 3.4$		$\frac{5\pi}{3}$	$-2\sqrt{3} \approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2} \approx 2.8$		$\frac{7\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{5\pi}{6}$	2		$\frac{11\pi}{6}$	-2
			2π	0



This is the graph of a circle. The equation r =4sinθ can be converted into rectangular coordinates by first multiplying both sides by r. This gives the equation r^2 =4rsinθ. Next use the facts that $r^2 = x^2+y^2$ and y=rsinθ. This gives $x^2+y^2 =4y$. To put this equation into standard form, subtract 4y from both sides of the equation and complete the square:

$x^2+y^2-4y = 0$

$x^2+(y^2-4y) = 0$

$x^2+(y^2-4y+4) = 0+4$

$x^2+(y-2)^2 = 4.$

This is the equation of a circle with radius 2 and center (0, 2) in the rectangular coordinate system.

Example 3.4

Transforming Polar Equations to Rectangular Coordinates

Rewrite each of the following equations in rectangular coordinates and identify the graph.

a. $\theta = \pi/3$

b. $r = 3$

c. $r = 6\cos\theta - 8\sin\theta$

Solution:

a. Take the tangent of both sides. This gives

$\tan\theta = \tan(\pi/3) =3.$

$\tan\theta = y/x$

$y/x = 3$

$y= x/3.$

This is the equation of a straight line passing through the origin with slope 3.

b. First, square both sides of the equation.

$r^2 =9$

$r^2 = x^2+y^2$

$x^2+y^2 =9,$

which is the equation of a circle centered at the origin with radius 3.

c. Multiply both sides of the equation by r

$r^2 =6r\cos\theta-8r\sin\theta.$

$r^2 = x^2+y^2,\ x =r\cos\theta,\ y=r\sin\theta.$

This gives

$r^2 = 6(r\cos\theta)-8(r\sin\theta)$

$x^2+y^2 = 6x-8y.$

To put this equation into standard form, first move the variables from the right-hand side of the equation to the left-hand side, then complete the square.

$x^2+y^2 = 6x-8y$

$x^2-6x+y^2+8y = 0$

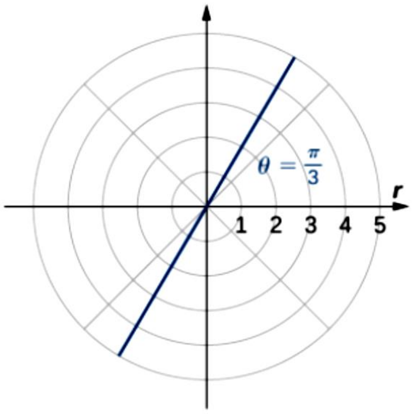
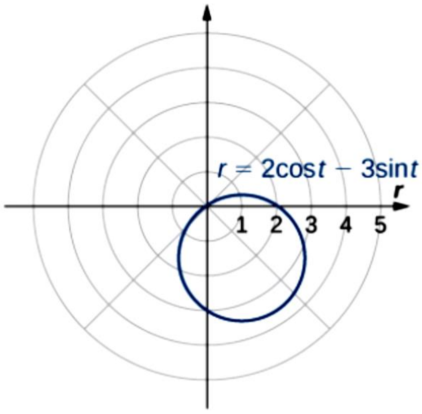
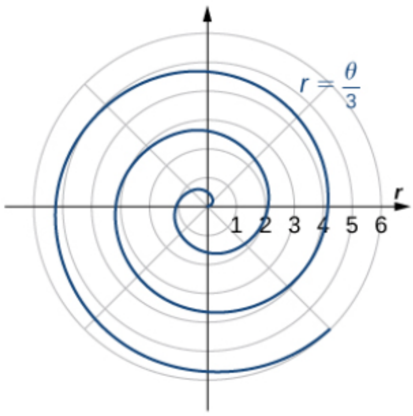
$(x^2-6x)+(y^2+8y) = 0$

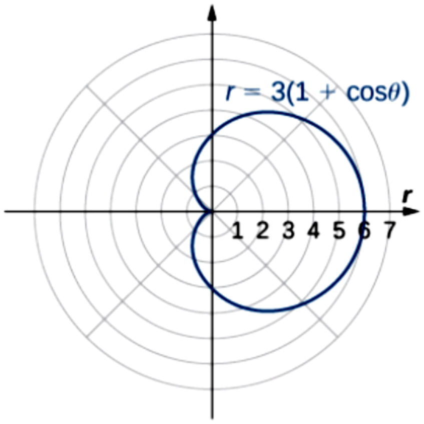
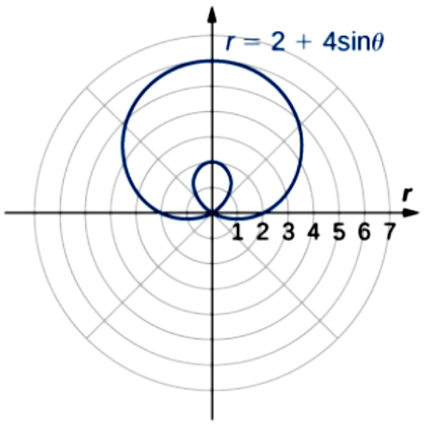
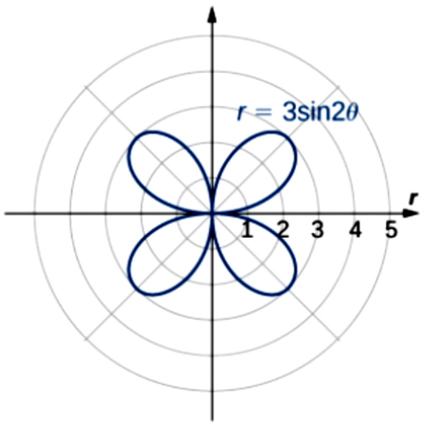
$(x^2-6x+9)+(y^2+8y+16) = 9+16$

$(x-3)^2+(y+4)^2 = 25.$

This is the equation of a circle with center at $(3,-4)$ and radius 5. Notice that the circle passes through the origin since the center is 5 units away.

We have now seen several examples of drawing graphs of curves defined by polar equations. A summary of some common curves is given in the tables below. In each equation, a and b are arbitrary constants.

Name	Equation	Example
Line passing through the pole with slope tan K	$\theta = K$	
Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	

Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	
Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	

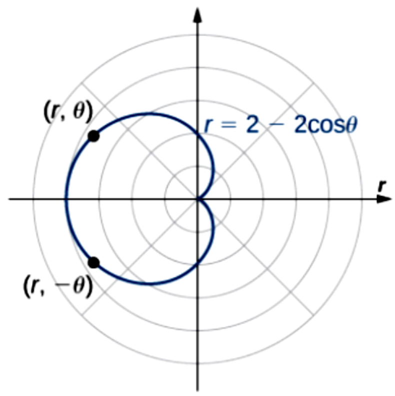
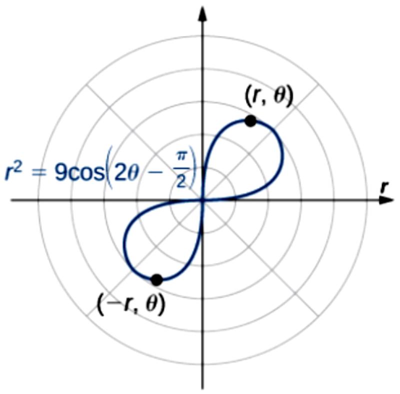
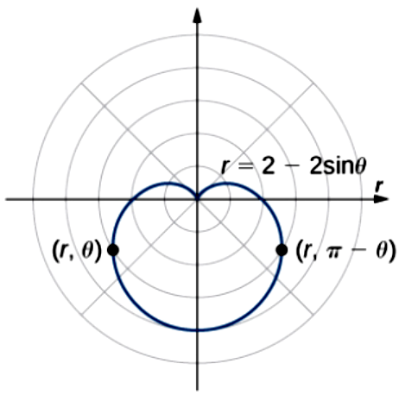
3.3 Symmetry in Polar Coordinates

Symmetry Tests for Polar Graphs in the Cartesian *xy*-Plane

1. Symmetry about the x-axis (polar): If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph.

2. Symmetry about the y-axis (vertical): If the point (r, θ) lies on the graph, then the point (r, π - θ) or (-r, - θ) lies on the graph.

3. Symmetry about the origin (pole): If the point (r, θ) lies on the graph, then the point (-r, θ) or (r, θ + π) lies on the graph.

Symmetry with respect to the polar axis: For every point (r, θ) on the graph, there is also a point reflected directly across the horizontal (polar) axis.	
Symmetry with respect to the pole: For every point (r, θ) on the graph, there is also a point on the graph that is reflected through the pole as well.	
Symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$: For every point (r, θ) on the graph, there is also a point reflected directly across the vertical axis.	

Example 3.5

Find the symmetry of the rose defined by the equation $r = 3\sin(2\theta)$ and create a graph.

Solution:

Suppose the point (r, θ) is on the graph of $r = 3\sin(2\theta)$.

i. To test for symmetry about the polar axis, first try replacing θ with $-\theta$. This gives $r = 3\sin(2(-\theta)) = -3\sin(2\theta)$. Since this changes the original equation, this test is not satisfied. However, returning to the original equation and replacing r with $-r$ and θ with $\pi-\theta$ yields

$-r = 3\sin(2(\pi-\theta))$

$-r = 3\sin(2\pi-2\theta)$

$-r = 3\sin(-2\theta)$

$-r = -3\sin 2\theta.$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta$, which is the original equation. This demonstrates that the graph is symmetric with respect to the polar axis.

ii. To test for symmetry with respect to the pole, first replace r with $-r$, which yields $-r = 3\sin(2\theta)$. Multiplying both sides by -1 gives $r = -3\sin(2\theta)$, which does not agree with the original equation. Therefore the equation does not pass the test for this symmetry. However, returning to the original equation and replacing θ with $\theta+\pi$ gives

$r = 3\sin(2(\theta+\pi))$

$= 3\sin(2\theta+2\pi)$

$= 3(\sin 2\theta\cos 2\pi+\cos 2\theta\sin 2\pi)$

$= 3\sin 2\theta.$

Since this agrees with the original equation, the graph is symmetric about the pole.

iii. To test for symmetry with respect to the vertical line $\theta = \pi/2$, first replace both r with $-r$ and θ with $-\theta$.

$-r = 3\sin(2(-\theta))$

$-r = 3\sin(-2\theta)$

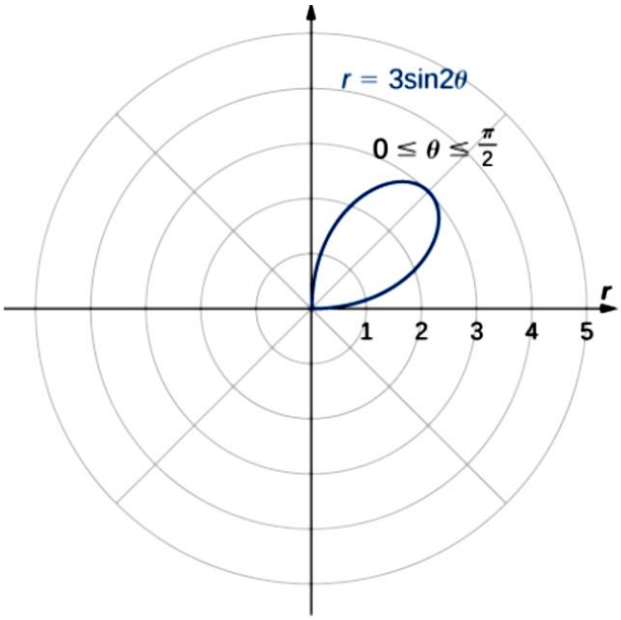
$-r = -3\sin 2\theta.$

Multiplying both sides of this equation by -1 gives $r = 3\sin 2\theta$, which is the original equation. Therefore the graph is symmetric about the vertical line $\theta = \pi/2$.

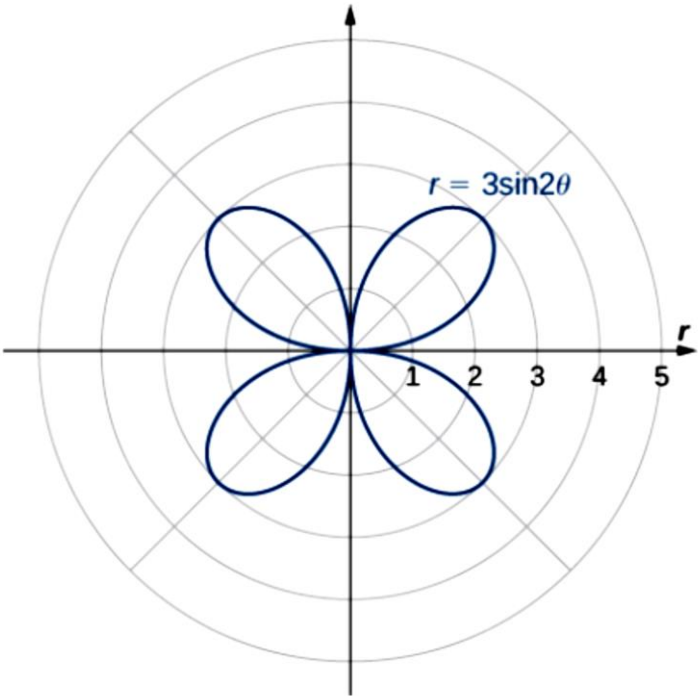
This graph has symmetry with respect to the polar axis, the origin, and the vertical line going through the pole. To graph the function, tabulate values of θ between 0 and $\pi/2$ and then reflect the resulting graph.

θ	r
0	0
$\frac{\pi}{6}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{4}$	3
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2} \approx 2.6$
$\frac{\pi}{2}$	0

This gives one petal of the rose, as shown in the following graph.

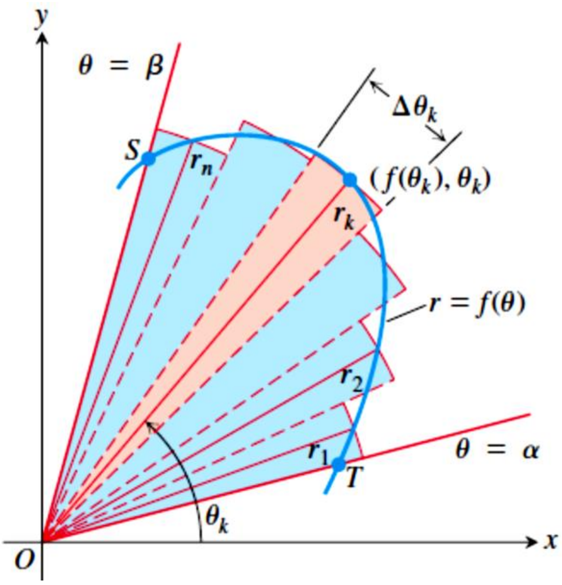


Reflecting this image into the other three quadrants gives the entire graph as shown.



3.4 Areas in Polar Coordinates

The region *OTS* in Figure 11.32 is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle TOS. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius r_k, or



$$A_k = \frac{1}{2}r_k{}^2\,\Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2\Delta\theta_k.$$

The area of region *OTS* is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n}\frac{1}{2}\left(f(\theta_k)\right)^2\Delta\theta_k.$$

If f is continuous, we expect the approximations to improve as the norm of the partition P goes to zero, where the norm of P is the largest value of $\Delta\theta_k$. We are therefore led to the following formula for the region's area:

$$A = \lim_{\|P\|\to 0}\sum_{k=1}^{n}\frac{1}{2}\left(f(\theta_k)\right)^2\Delta\theta_k = \int_{\alpha}^{\beta}\frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

Example 3.7

Find the area of the region in the xy-plane enclosed by the cardioid r = 2(1 + cos θ).

Solution:

We graph the cardioid (Figure) and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

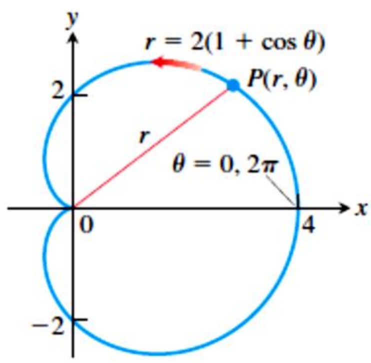
$$\int_{\theta=0}^{\theta=2\pi}\frac{1}{2}r^2\,d\theta = \int_0^{2\pi}\frac{1}{2}\cdot 4(1+\cos\theta)^2\,d\theta$$

$$= \int_0^{2\pi} 2(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= \int_0^{2\pi}\left(2+4\cos\theta+2\cdot\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \int_0^{2\pi}(3+4\cos\theta+\cos 2\theta)\,d\theta$$

$$= \left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi.$$

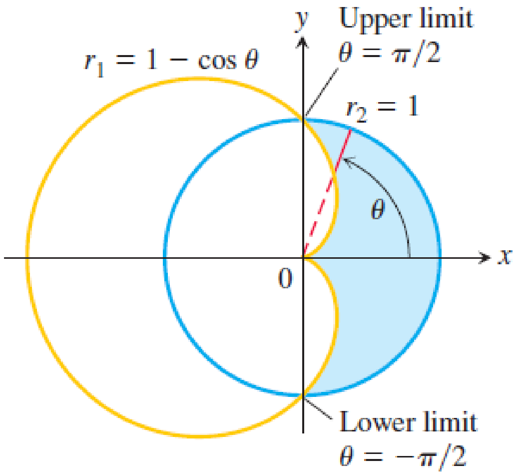


Example 3.8

Find the area of the region that lies inside the circle r = 1 and outside the cardioid r = 1 - cos θ.

Solution:

We sketch the region to determine its boundaries and find the limits of integration (Figure). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. The area, from Equation (1), is



$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Eq. (1)}$$

$$= 2\int_{0}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry}$$

$$= \int_{0}^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \qquad r_2 = 1 \text{ and } r_1 = 1 - \cos\theta$$

$$= \int_{0}^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_{0}^{\pi/2}\left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}.$$

3.5 Arc Length of a Curve Defined by a Polar Function

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of r = f (θ) from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Example 3.9

Find the arc length of the cardioid r = 2 + 2cos θ .

Solution

When θ = 0, r = 2 + 2cos θ = 4. Furthermore, as θ goes from 0 to 2π, the cardioid is traced out exactly once. Therefore these are the limits of integration. Using f (θ) = 2 + 2cos θ , α = 0, and $\beta = 2\pi$,

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{[2+2\cos\theta]^2 + [-2\sin\theta]^2}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{4 + 8\cos\theta + 4(\cos^2\theta + \sin^2\theta)}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta \\ &= 2\int_{0}^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta. \end{aligned}$$

Next, using the identity $\cos(2\alpha) = 2\cos^2\alpha - 1$, add 1 to both sides and multiply by 2. This gives $2+2\cos(2\alpha)=4\cos^2\alpha$. Substituting $\alpha = \theta/2$ gives $2 + 2\cos\theta = 4\cos^2(\theta/2)$, so the integral becomes

$$\begin{aligned} L &= 2\int_{0}^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\ &= 2\int_{0}^{2\pi} \sqrt{4\cos^2\left(\frac{\theta}{2}\right)}\, d\theta \\ &= 2\int_{0}^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta. \end{aligned}$$

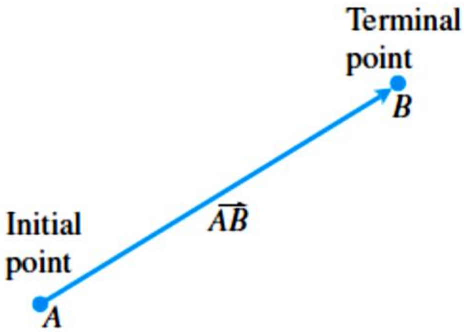
The absolute value is necessary because the cosine is negative for some values in its domain. To resolve this issue، change the limits from 0 to π and double the answer. This strategy works because cosine is positive between 0 and π/2. Thus,

$$\begin{aligned} L &= 4\int_{0}^{2\pi} \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta \\ &= 8\int_{0}^{\pi} \cos\left(\frac{\theta}{2}\right) d\theta \\ &= 8\left(2\sin\left(\frac{\theta}{2}\right)\right|_{0}^{\pi} \\ &= 16. \end{aligned}$$

4. Vectors and the Geometry of Space

4.1 Vectors

A quantity such as force, displacement, or velocity is called a vector and is represented by a directed line segment (as shown in Figure). The vector represented by the directed line segment $\overrightarrow{AB}$ has initial point A and terminal point B and its length is denoted by $|\overrightarrow{AB}|$. Two vectors are equal if they have the same length and direction.



If v is a two-dimensional vector in the plane equal to the vector with initial point at the origin and terminal point (v1, v2), then the component form of v is

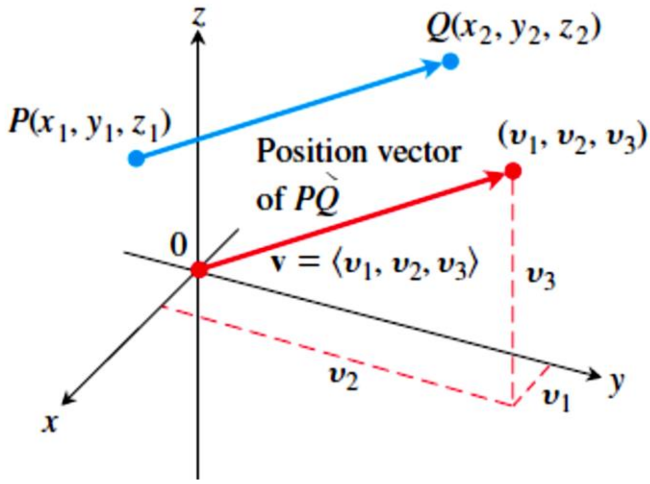
v = (v1, v2).

If v is a three-dimensional vector equal to the vector with initial point at the origin and terminal point (v1, v2, v3), then the component form of v is

v = (v1, v2, v3).

The magnitude or length of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Example:

Find the **(a)** component form and **(b)** length of the vector with initial point *P*(-3, 4, 1) and terminal point *Q*(-5, 2, 2).

Solution:

(a) The standard position vector **v** representing $\overrightarrow{PQ}$ has components

$$v_1 = x_2 - x_1 = -5 - (-3) = -2, \qquad v_2 = y_2 - y_1 = 2 - 4 = -2,$$

and

$$v_3 = z_2 - z_1 = 2 - 1 = 1.$$

The component form of $\overrightarrow{PQ}$ is

$$\mathbf{v} = \langle -2, -2, 1 \rangle.$$

(b) The length or magnitude of $\mathbf{v} = \overrightarrow{PQ}$ is

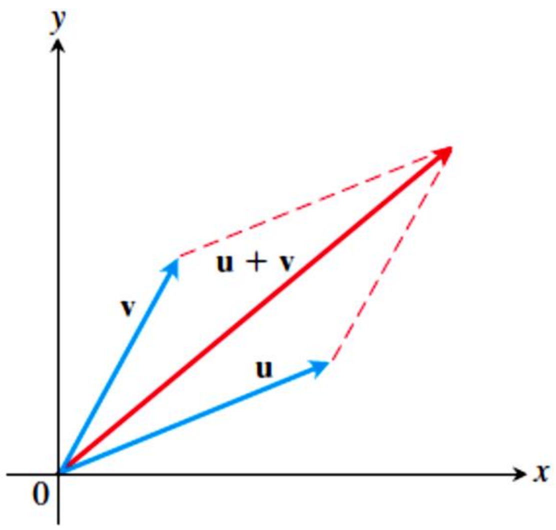
$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

4.1.1 Vector Algebra Operations

Let u = (u1, u2, u3) and v = (v1, v2, v3) be vectors with k a scalar

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$



Example:

Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find the components of

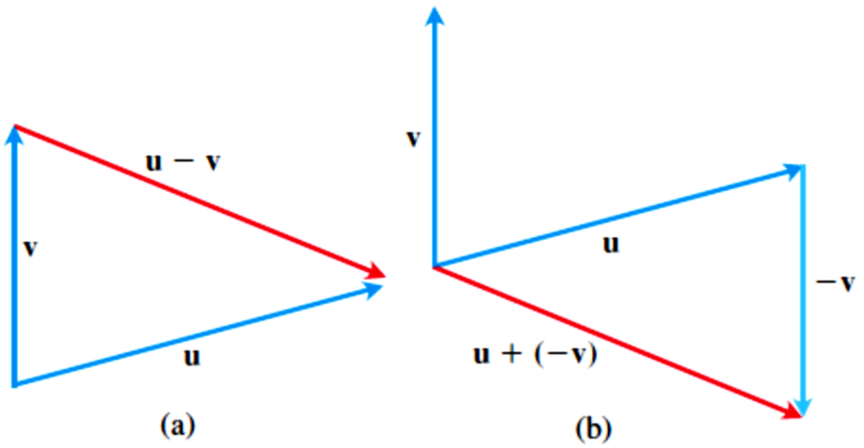
(a) $2\mathbf{u} + 3\mathbf{v}$ **(b)** $\mathbf{u} - \mathbf{v}$ **(c)** $\left|\frac{1}{2}\mathbf{u}\right|$.

Solution:

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $\left|\frac{1}{2}\mathbf{u}\right| = \left|\left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$



4.1.2 Properties of Vector Operations

Let **u**, **v**, **w** be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

4.1.2 Unit Vectors

A vector v of length 1 is called a unit vector. The standard unit vectors are

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a linear combination of the standard unit vectors as follows:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle$$
$$= v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle$$
$$= v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}.$$

The vector from P1(x1, y1, z1) to P2(x2, y2, z2) is

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

Example:

Find a unit vector **u** in the direction of the vector from $P_1(1, 0, 1)$ to

$P_2(3, 2, 0)$.

Solution:

We divide $\overrightarrow{P_1P_2}$ by its length:

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$
$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$
$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

This unit vector **u** is the direction of $\overrightarrow{P_1P_2}$.

Example:

If **v** = 3**i** - 4**j** is a velocity vector, express **v** as a product of its speed times its direction of motion.

Solution:

Speed is the magnitude (length) of **v**:

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector v> 0 v 0 is the direction of v:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = \underbrace{5}_{\text{Length (speed)}}\underbrace{\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)}_{\text{Direction of motion}}.$$

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector called the direction of $\mathbf{v}$;

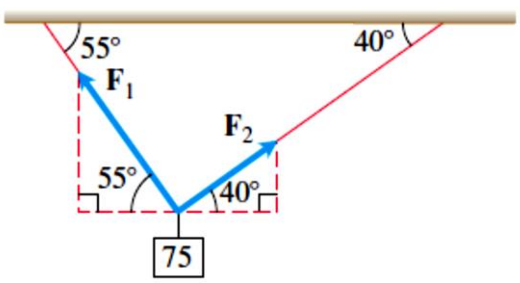
2. the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ as its length times its direction.

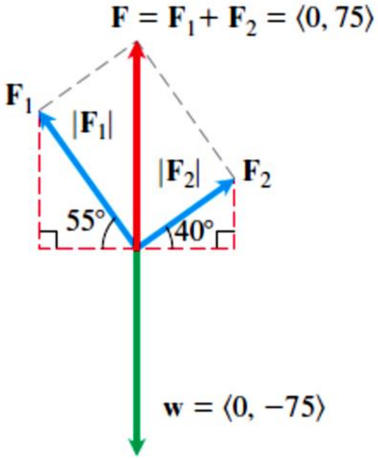
The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Example:

A 75-N weight is suspended by two wires, as shown in Figure. Find the forces **F**1 and **F**2 acting in both wires.





Solution:

$$\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 55°,\ |\mathbf{F}_1|\sin 55° \rangle \qquad \text{and} \qquad \mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 40°,\ |\mathbf{F}_2|\sin 40° \rangle.$$

Since $\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 75 \rangle$, the resultant vector leads to the system of equations

$$\begin{aligned} -|\mathbf{F}_1|\cos 55° + |\mathbf{F}_2|\cos 40° &= 0 \\ |\mathbf{F}_1|\sin 55° + |\mathbf{F}_2|\sin 40° &= 75. \end{aligned}$$

Solving for $|\mathbf{F}_2|$ in the first equation and substituting the result into the second equation, we get

$$|\mathbf{F}_2| = \frac{|\mathbf{F}_1|\cos 55°}{\cos 40°} \qquad \text{and} \qquad |\mathbf{F}_1|\sin 55° + \frac{|\mathbf{F}_1|\cos 55°}{\cos 40°}\sin 40° = 75.$$

It follows that

$$|\mathbf{F}_1| = \frac{75}{\sin 55° + \cos 55° \tan 40°} \approx 57.67 \text{ N},$$

and

$$\begin{aligned} |\mathbf{F}_2| &= \frac{75 \cos 55°}{\sin 55° \cos 40° + \cos 55° \sin 40°} \\ &= \frac{75 \cos 55°}{\sin (55° + 40°)} \approx 43.18 \text{ N}. \end{aligned}$$

The force vectors are then

$$\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 55°, |\mathbf{F}_1|\sin 55° \rangle \approx \langle -33.08, 47.24 \rangle$$

and

$$\mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 40°, |\mathbf{F}_2|\sin 40° \rangle \approx \langle 33.08, 27.76 \rangle.$$

4.2 The Dot Product

The **dot product u . v** (**"u dot v"**) of vectors **u** = ($u1$, $u2$, $u3$) and **v** = ($v1$, $v2$, $v3$) is the scalar

u . v = u1v1 + u2v2 + u3v3.

Example:

(a) $\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3)$

$$= -6 - 4 + 3 = -7$$

(b) $\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$

4.2.1 Dot Product and Angles

The angle between two nonzero vectors u and v is

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right).$$

The dot product of two vectors u and v is given by

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta.$$

Example:

Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution:

$$\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \approx 1.76 \text{ radians or } 100.98°.$$

Example:

Find the angle u in the triangle *ABC* determined by the vertices

$A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$.

Solution:

The angle u is the angle between the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. The component forms of these two vectors are

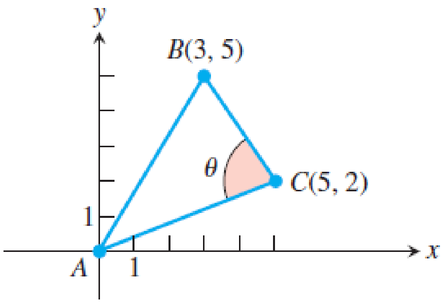
$$\overrightarrow{CA} = \langle -5, -2 \rangle \quad \text{and} \quad \overrightarrow{CB} = \langle -2, 3 \rangle.$$

First we calculate the dot product and magnitudes of these two vectors.

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4$$
$$|\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$
$$|\overrightarrow{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

Then applying the angle formula, we have

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}\right) = \cos^{-1}\left(\frac{4}{(\sqrt{29})(\sqrt{13})}\right)$$
$$\approx 78.1° \text{ or } 1.36 \text{ radians.}$$



4.2.2 Properties of the Dot Product

If **u**, **v**, and **w** are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0.$

Orthogonal Vectors: Vectors u and v are orthogonal if

u . v = 0.

The vector projection of **u** onto **v** is the vector

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|}.$$

The scalar component of **u** in the direction of **v** is the scalar

$$|\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}.$$

Work: The **work** done by a constant force **F** acting through a displacement **D** = $\overrightarrow{PQ}$ is

W = **F . D**

Example:

Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto

$\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$.

Solution:

We find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$:

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{6-6-4}{1+4+4}(\mathbf{i}-2\mathbf{j}-2\mathbf{k})$$

$$= -\frac{4}{9}(\mathbf{i}-2\mathbf{j}-2\mathbf{k}) = -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.$$

We find the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$:

$$|\mathbf{u}|\cos\theta = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i}+3\mathbf{j}+2\mathbf{k})\cdot\left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right)$$

$$= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.$$

Example:

Verify that the vector $\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to the projection vector $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

Solution:

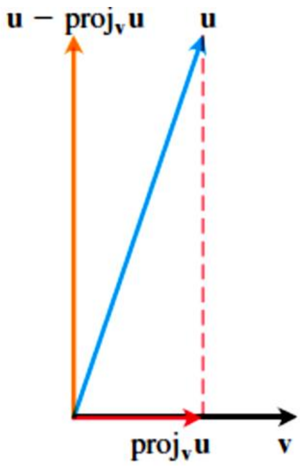
The vector $\text{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$ is parallel to $\mathbf{v}$. So it suffices to show that the vector $\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$ is orthogonal to $\mathbf{v}$. We verify orthogonality by showing that the dot product of $\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u}$ with $\mathbf{v}$ is zero:

$$(\mathbf{u} - \text{proj}_{\mathbf{v}}\,\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\right)\cdot\mathbf{v} \qquad \text{Definition of } \text{proj}_{\mathbf{v}}\,\mathbf{u}$$

$$= \mathbf{u}\cdot\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}\cdot\mathbf{v} \qquad \text{Dot product property (2)}$$

$$= \mathbf{u}\cdot\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}|\mathbf{v}|^2 \qquad \mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$$

$$= \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} = 0.$$



Example:

If $|\mathbf{F}| = 40$ N (newtons), $|\mathbf{D}| = 3$ m, and $\theta = 60°$, the work done by $\mathbf{F}$ in acting from P to Q is

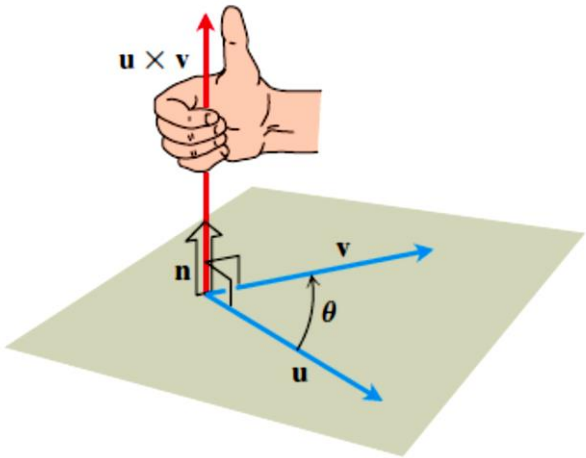
Solution:

$$\begin{aligned}\text{Work} &= \mathbf{F}\cdot\mathbf{D} && \text{Definition}\\ &= |\mathbf{F}||\mathbf{D}|\cos\theta \\ &= (40)(3)\cos 60° && \text{Given values}\\ &= (120)(1/2) = 60 \text{ J (joules).}\end{aligned}$$

4.3 The Cross Product

4.3.1 The Cross Product of Two Vectors in Space

We start with two nonzero vectors u and v in space. Two vectors are parallel if one is a nonzero multiple of the other. If u and v are not parallel, they determine a plane. The vectors in this plane are linear combinations of u and v, so they can be written as a sum au + bv. We select the unit vector n perpendicular to the plane by **the right-hand rule**. This means that we choose n to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle u from u to v (Figure). Then we define a new vector as follows.



DEFINITION The **cross product u × v ("u cross v")** is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\, \mathbf{n}.$$

Parallel Vectors

Nonzero vectors u and v are parallel if and only if u X v = 0.

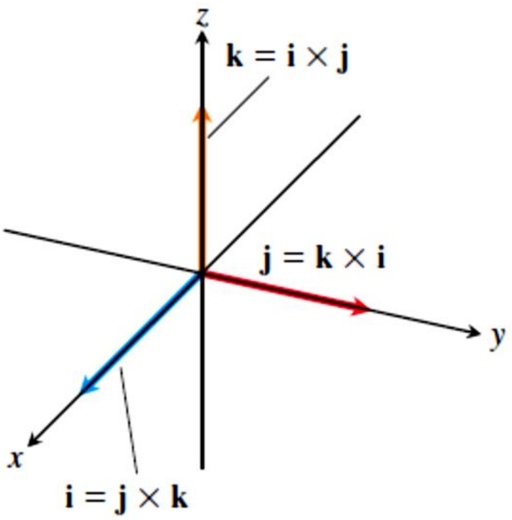
Properties of the Cross Product

If u, v, and w are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$$

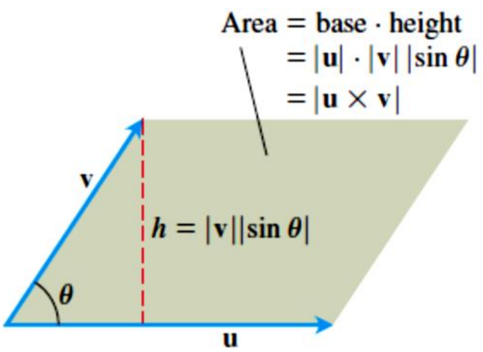
$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$



The Cross Product Is the Area of a Parallelogram:

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\,|\sin\theta|\,|\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin\theta.$$

Where n is a unit vector



4.3.2 Determinant Formula for u X v

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

<u>Example:</u>

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Solution:

We expand the symbolic determinant:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k} \qquad \text{Property 3}$$

Example:

Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure).

Solution:

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$

$$= 6\mathbf{i} + 6\mathbf{k}.$$

Example:

Find the area of the triangle with vertices P(1, -1, 0), Q(2, 1, -1), and R(-1, 1, 2) (Figure 12.32).

Solution:

The area of the parallelogram determined by P, Q, and R is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6\mathbf{i} + 6\mathbf{k}|$$

$$= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}.$$

The triangle's area is half of this, or $3\sqrt{2}$.

Example:

Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution:

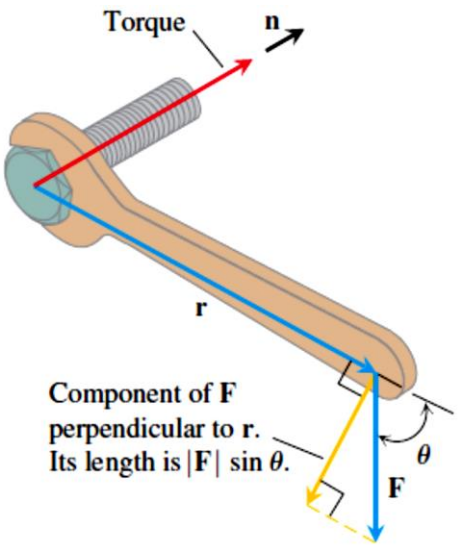
Solution Since $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane, its direction $\mathbf{n}$ is a unit vector perpendicular to the plane. Taking values from Examples 2 and 3, we have

$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

4.3.3 Torque

$$\text{Magnitude of torque vector} = |\mathbf{r}||\mathbf{F}|\sin\theta,$$

$$\text{Torque vector} = \mathbf{r} \times \mathbf{F} = (|\mathbf{r}||\mathbf{F}|\sin\theta)\,\mathbf{n}.$$



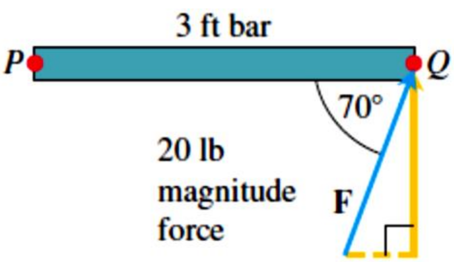
Example:

Find the magnitude of the torque generated by force F at the pivot point P in Figure.

Solution:

$$|\overrightarrow{PQ} \times \mathbf{F}| = |\overrightarrow{PQ}||\mathbf{F}|\sin 70° \approx (3)(20)(0.94) \approx 56.4 \text{ ft-lb}.$$

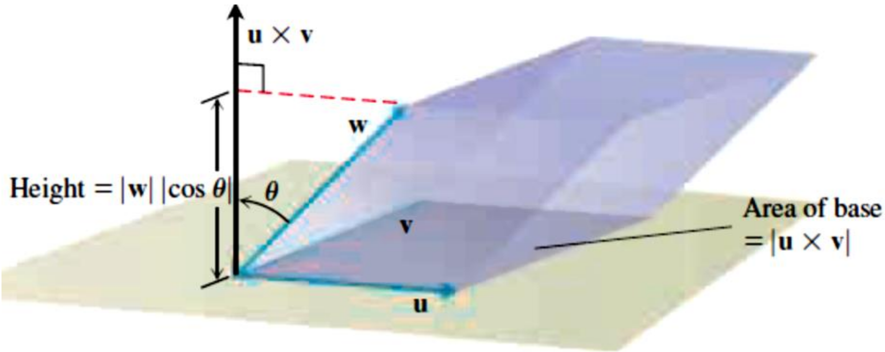
In this example the torque vector is pointing out of the page toward you.



4.3.4 Triple Scalar or Box Product

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



Volume = area of base · height
= |u × v| |w| |cos θ|
= |(u × v) · w|

Example:

Find the volume of the box (parallelepiped) determined by

u = i + 2j - k, v = -2i + 3k, and w = 7j - 4k.

Solution:

Using the rule for calculating a 3 * 3 determinant, we find

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = (1)\begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - (2)\begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed.

4.4 Lines and Planes in Space

4.4.1 Lines and Line Segments in Space

Vector Equation for a Line:

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to v is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty,$$

Where r is the position vector of a point $P(x, y, z)$ on L and r_0 is the position vector of $P_0(x_0, y_0, z_0)$.

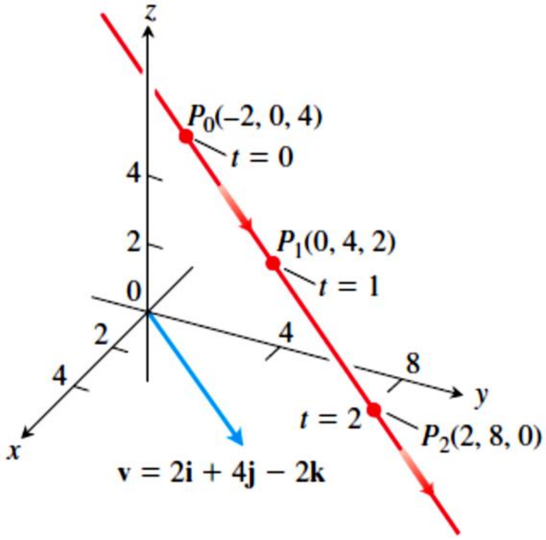
Parametric Equations for a Line:

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to v = y_1 i + y_2 j + y_3 k is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty$$

Example:

Find parametric equations for the line through (-2, 0, 4) parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (Figure).



Solution:

With $P_0(x_0, y_0, z_0)$ equal to (-2, 0, 4) and $v1\,\mathbf{i} + v2\,\mathbf{j} + v3\,\mathbf{k}$ equal to $2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, Equations (3) become

$x = -2 + 2t, \; y = 4t, \; z = 4 - 2t.$

Example:

Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution:

The vector

$$\overrightarrow{PQ} = (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$$

is parallel to the line, and Equations (3) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$x = -3 + 4t, y = 2 - 3t, z = -3 + 7t$.

We could have chosen $Q(1, -1, 4)$ as the “base point” and written

$x = 1 + 4t, \; y = -1 - 3t, \; z = 4 + 7t$.

These equations serve as well as the first; they simply place you at a different point on the line for a given value of t.

Example:

Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (Figure).

Solution:

We begin with equations for the line through P and Q, taking them, in this case, from Example 2:

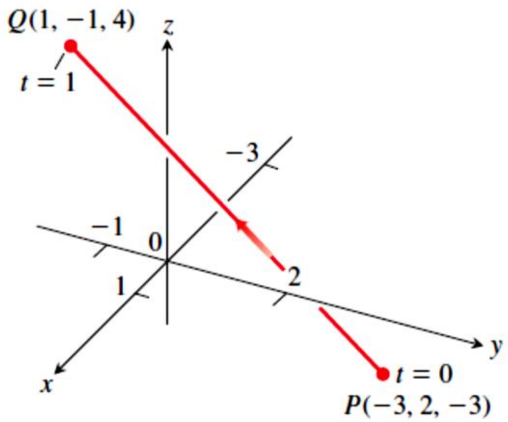
$x = -3 + 4t,\ \ y = 2 - 3t,\ \ z = -3 + 7t$.

We observe that the point

$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$

on the line passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \le t \le 1$ to parametrize the segment:

$x = -3 + 4t, \quad y = 2 - 3t, \quad z = -3 + 7t, \quad 0 \le t \le 1.$



$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

$$= \underbrace{\mathbf{r}_0}_{\text{Initial position}} + \underbrace{t}_{\text{Time}}\underbrace{|\mathbf{v}|}_{\text{Speed}}\underbrace{\frac{\mathbf{v}}{|\mathbf{v}|}}_{\text{Direction}}.$$

Example:

A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

Solution:

We place the origin (0,0,0) at the starting position (helipad) of the helicopter. Then the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$$

gives the flight direction of the helicopter. From Equation (4), the position of the helicopter at any time t is

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}_0 + t(\text{speed})\mathbf{u} \\ &= \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) \\ &= 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).\end{aligned}$$

When $t = 10$ sec,

$$\begin{aligned}\mathbf{r}(10) &= 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3}\rangle .\end{aligned}$$

After 10 sec of flight from the origin toward (1, 1, 1), the helicopter is located at the point $(200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3})$ in space. It has traveled a distance of (60 ft/sec)(10 sec) = 600 ft, which is the length of the vector $\mathbf{r}(10)$. ■

4.4.2 The Distance from a Point to a Line in Space

Distance from a point S to a line through P parallel to v

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|}$$

Example:

Find the distance from the point $S(1, 1, 5)$ to the line

L: $x = 1 + t$, $y = 3 - t$, $z = 2t$.

Solution:

We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to

$\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

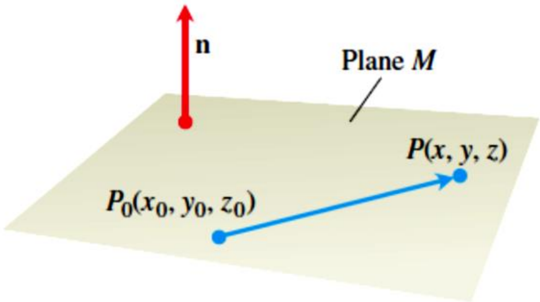
4.4.3 An Equation for a Plane in Space

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\,\mathbf{k}$ has

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$



Example:

Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to

$\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution:

The component equation is

$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$

Simplifying, we obtain

$5x + 15 + 2y - z + 7 = 0$

$5x + 2y - z = -22.$

Notice in Example how the components of $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ became the coefficients of x, y, and z in the equation $5x + 2y - z = -22$. The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\,\mathbf{k}$ is normal to the plane $Ax + By + Cz = D$.

<u>Example:</u>

Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

Solution:

We find a vector normal to the plane and use it with one of the points (it does not matter which) to write an equation for the plane.

The cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

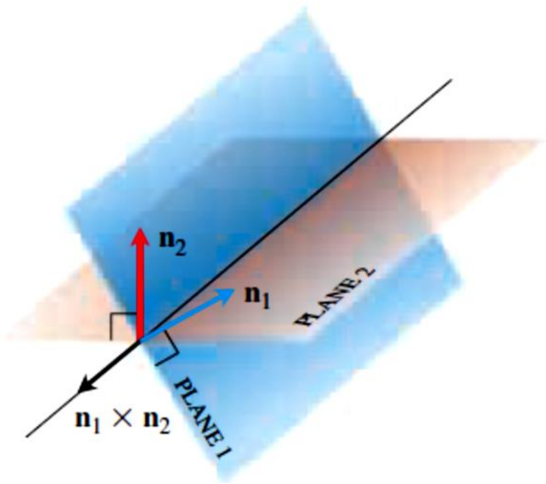
is normal to the plane. We substitute the components of this vector and the coordinates of $A(0, 0, 1)$ into the component form of the equation to obtain

$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$

$3x + 2y + 6z = 6.$

4.4.4 Lines of Intersection

Just as lines are parallel if and only if they have the same direction, two planes are **parallel** if and only if their normals are parallel, or $\mathbf{n}1 = k\mathbf{n}2$ for some scalar k. Two planes that are not parallel intersect in a line.



Example:

Find a vector parallel to the line of intersection of the planes

$3x - 6y - 2z = 15$ and $2x + y - 2z = 5$, and find parametric equations for the line.

Solution:

1) The line of intersection of two planes is perpendicular to both planes' normal vectors $\mathbf{n}1$ and $\mathbf{n}_2$ (Figure) and therefore parallel to $\mathbf{n}1 \times \mathbf{n}_2$. Turning this around, $\mathbf{n}_1 \times \mathbf{n}_2$ is a vector parallel to the planes' line of intersection. In our case,

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}.$$

Any nonzero scalar multiple of $\mathbf{n}_1 \times \mathbf{n}_2$ will do as well.

2) $\mathbf{v} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$ as a vector parallel to the line. To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for x and y simultaneously identiies one of these points as $(3, -1, 0)$. The line is

$x = 3 + 14t,\ y = -1 + 2t,\ z = 15t.$

The choice $z = 0$ is arbitrary and we could have chosen $z = 1$ or $z = -1$ just as well. Or we could have let $x = 0$ and solved for y and z. The different choices would simply give different parametrizations of the same line.

Example:

Find the point where the line

$x = 8/3 + 2t, \quad y = -2t, \quad z = 1 + t$

intersects the plane $3x + 2y + 6z = 6$.

Solution:

The point

$$\left(\frac{8}{3} + 2t, -2t, 1 + t\right)$$

lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8$$

$$t = -1.$$

The point of intersection is

$$(x, y, z)\big|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right).$$

4.4.5 The Distance from a Point to a Plane

Distance from a Point S to a Plane with Normal $\mathbf{n}$ at Point P

$$d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Example:

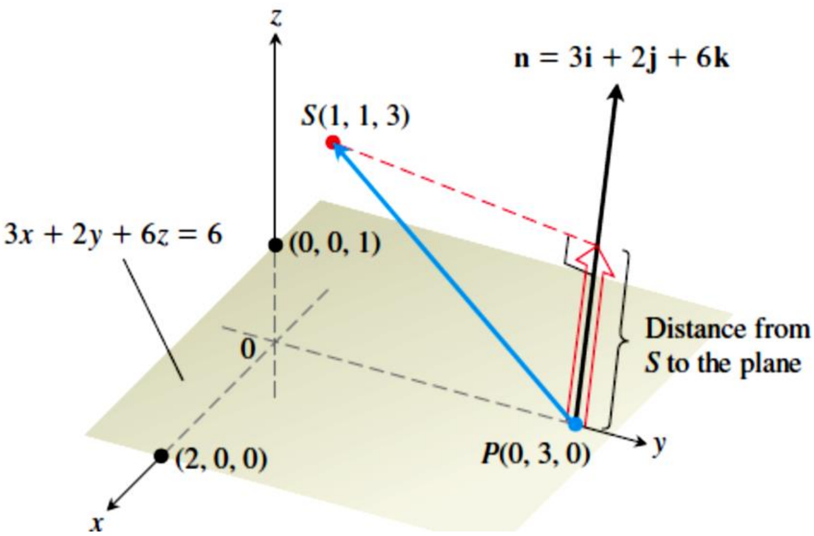
Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Solution:

We find a point P in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector $\mathbf{n}$ normal to the plane (Figure). The coefficients in the equation

$3x + 2y + 6z = 6$ give

$\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$.



The points on the plane easiest to find from the plane's equation are the intercepts. If we take P to be the y-intercept $(0, 3, 0)$, then

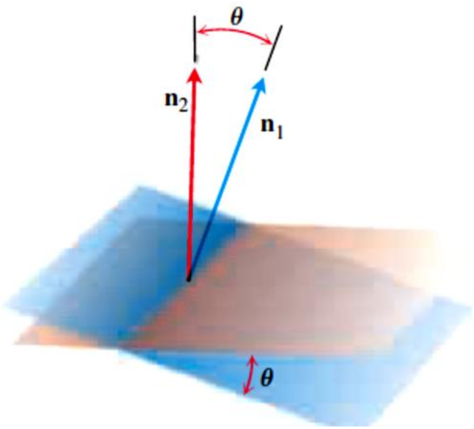
$$\overrightarrow{PS} = (1-0)\mathbf{i} + (1-3)\mathbf{j} + (3-0)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},$$
$$|\mathbf{n}| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{49} = 7.$$

Therefore, the distance from S to the plane is

$$d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| \qquad \text{Length of } \operatorname{proj}_{\mathbf{n}} \overrightarrow{PS}$$
$$= \left|(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \cdot \left(\frac{3}{7}\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)\right|$$
$$= \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \frac{17}{7}.$$

4.4.6 Angles Between Planes

The angle between two intersecting planes is defined to be the acute angle between their normal vectors (Figure).



Example:

Find the angle between the planes $3x - 6y - 2z = 15$ and

$2x + y - 2z = 5$.

Solution:

The vectors

$\mathbf{n}1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$, $\mathbf{n}2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

are normals to the planes. The angle between them is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right)$$

$$= \cos^{-1}\left(\frac{4}{21}\right) \approx 1.38 \text{ radians.} \qquad \text{About 79 degrees}$$